CHAPTER 21

The Riemann–Roch Theorem

21a. Spaces of Functions and 1-Forms

Fix a compact Riemann surface X, and let $g = g_X$ be its genus, M its field of meromorphic functions, and Ω the space of meromorphic 1-forms on X. A *divisor* $D = \Sigma m_P P$ on X is just another word for a 0-chain. That is, it assigns an integer m_P to each point P in X, with only finitely many being nonzero. We say that the *order* of D at P is m_P, and write $\mathrm{ord}_P(D) = m_P$. The divisors on X form an abelian group. As for 0-chains, the *degree* of a divisor is the sum of the coefficients: $\deg(D) = \Sigma m_P$. If $E = \Sigma n_P P$ is another divisor, we write $E \geq D$ to mean that $n_P \geq m_P$ for all P in X. A divisor D is called *effective* if each coefficient m_P is nonnegative, i.e., $D \geq 0$.

Any nonzero meromorphic function f on X determines a divisor

$$\mathrm{Div}(f) \;=\; \Sigma\, \mathrm{ord}_P(f)\, P\,.$$

Similarly, any nonzero meromorphic 1-form ω on X determines a divisor

$$\mathrm{Div}(\omega) \;=\; \Sigma\, \mathrm{ord}_P(\omega) P\,.$$

Corollary 19.5 and Proposition 20.14 say that

$$\deg(\mathrm{Div}(f)) \;=\; 0 \quad \text{and} \quad \deg(\mathrm{Div}(\omega)) \;=\; 2g-2\,.$$

Our goal in this chapter is to find meromorphic functions and 1-forms with prescribed, or at least controlled behavior. For example, we want to find functions with poles only at certain points, and with the orders of poles at these points not exceeding some bounds. For a divisor $D = \Sigma m_P P$ on X let

$$\begin{aligned} L(D) &= \{f \in M\colon \mathrm{ord}_P(f) \geq -m_P \text{ for all } P \in X\} \\ &= \{f \in M\colon \mathrm{Div}(f) + D \geq 0\}. \end{aligned}$$

This set of functions $L(D)$ is a complex subspace of M. Similarly, let

$$\begin{aligned} \Omega(D) &= \{\omega \in \Omega\colon \mathrm{ord}_P(\omega) \geq m_P \text{ for all } P \in X\} \\ &= \{\omega \in \Omega\colon \mathrm{Div}(\omega) \geq D\}, \end{aligned}$$

a complex subspace of Ω. For example, $\Omega(0)$ is the space $\Omega^{1,0}$ of holomorphic 1-forms on X. Note that $L(D)$ allows poles at the points P where $m_P > 0$, while $\Omega(D)$ requires zeros at the same points.

Lemma 21.1. *(a) $L(D) = 0$ if $deg(D) < 0$, and $\Omega(D) = 0$ if $deg(D) > 2g - 2$.*

(b) For any D and any point Q in X, $L(D) \subset L(D+Q)$, and $\Omega(D) \supset \Omega(D+Q)$. In addition,

$$\dim(L(D+Q)/L(D)) \leq 1 \quad \textit{and} \quad \dim(\Omega(D)/\Omega(D+Q)) \leq 1.$$

(c) $L(D)$ and $\Omega(D)$ are finite-dimensional vector spaces.

Proof. (a) follows from the fact that $\deg(\mathrm{Div}(f)) = 0$ and $\deg(\mathrm{Div}(\omega)) = 2g - 2$. To prove (b), fix a local coordinate function z at Q, and let $m = \mathrm{ord}_Q(D)$. Any f in $L(D+Q)$ has a local expression $h(z)/z^{m+1}$, with h holomorphic at 0. The map which assigns $h(0)$ to f determines a homomorphism of complex vector spaces from $L(D+Q)$ to $\mathbb{C}$, whose kernel is exactly $L(D)$. This shows that either $L(D) = L(D+Q)$ or the quotient $L(D+Q)/L(D)$ is one dimensional. Similarly, any ω in $\Omega(D)$ has a local expression $h(z)z^m\,dz$ for h holomorphic, and assigning $h(0)$ to ω determines a map from $\Omega(D)$ to $\mathbb{C}$ whose kernel is $\Omega(D+Q)$.

It follows from (b) that $L(D)$ is finite dimensional if and only if $L(D+Q)$ is finite dimensional. Since one can get from any D to a divisor of negative degree by a finite number of subtractions of a point, the fact that $L(D)$ is finite dimensional follows from (a). Similarly for $\Omega(D)$, one can add points until the degree gets larger than $2g - 2$. □

Exercise 21.2. If $D \leq E$, show that

$$\dim(L(E)) - \dim(L(D)) \leq \deg(E) - \deg(D).$$

One sees from the preceding proof that $\dim(L(D)) \leq \deg(D) + 1$.

For example, $L(0)=\mathbb{C}$ has dimension 1. If $D=Q$ is a point, however, we see from Exercise 20.7 that $L(Q)$ is also $\mathbb{C}$ unless $X\cong S^2$.

Exercise 21.3. If $X=S^2$, show that $\dim(L(D))=\deg(D)+1$ whenever $\deg(D)\geq -1$.

Lemma 21.4. (*a*) *For any nonzero meromorphic function f and any divisor D,*

$$\dim(L(D))=\dim(L(D+\mathrm{Div}(f))).$$

(*b*) *For any nonzero meromorphic* 1*-form* ω *and any divisor D,*

$$\dim(\Omega(D))=\dim(L(\mathrm{Div}(\omega)-D)).$$

Proof. We have isomorphisms

$$L(D)\rightarrow L(D+\mathrm{Div}(f)),\qquad h\mapsto h\cdot f,$$

and

$$L(\mathrm{Div}(\omega)-D)\rightarrow\Omega(D),\qquad h\mapsto h\cdot\omega,$$

from which the lemma follows. □

Although it has been fairly easy to get an upper bound for the size of $L(D)$, it is not so easy to get lower bounds, i.e., to show that there must be functions with given poles. When X comes from an algebraic curve, however, Proposition 20.8 gives a first step, for at least one divisor. Take $z\colon X\rightarrow S^2$ as in that proposition, and let E be the divisor of *poles* of z, that is,

$$E=\sum_{z(P)=\infty}e_z(P)P.$$

This is a divisor of degree n on X, where n is the degree of the mapping z.

Lemma 21.5. *For this divisor E, there is a constant k such that for all integers m,*

$$\dim(L(mE))\geq\deg(mE)+1-k=mn+1-k.$$

Proof. We need the following fact: for any meromorphic function h on X, there is a nonzero polynomial $p(z)$ in $\mathbb{C}[z]$ and an integer t so that $p(z)\cdot h$ is in $L(tE)$. To prove this, we need only find $p(z)$ so that $p(z)\cdot h$ has no poles outside E, and one sees that $\Pi(z-z(P))^{-\mathrm{ord}_P(h)}$, the product over all P such that $z(P)\neq\infty$ and $\mathrm{ord}_P(h)<0$, is such a polynomial.

We saw in Proposition 20.8 (see Lemma C.19) that M is a vector space over $\mathbb{C}(z)$ of dimension n. By the fact proved in the preceding paragraph, we can find a basis $h_1, \ldots, h_n$ for M over $\mathbb{C}(z)$ and an integer t so that each h_i is in $L(tE)$. Now for $m = t + s$, $s \geq 0$, the $(s+1) \cdot n$ functions $z^j \cdot h_i$, $0 \leq j \leq s$, $1 \leq i \leq n$, are all in $L(mE)$. This means that, for such m, the dimension of $L(mE)$ is at least $(m - t + 1) \cdot n = mn + 1 - k$ for some constant k. Increasing k if necessary, one may also achieve this inequality for the finite number of m with $0 \leq m < t$. The inequality is automatic for $m < 0$ and any $k \geq 0$, so the lemma is proved. $\square$

Lemma 21.6. *There are integers k and N so that*

$$\dim(L(D)) \;\geq\; \deg(D) + 1 - k$$

for all divisors D on X, with equality

$$\dim(L(D)) \;=\; \deg(D) + 1 - k \qquad \textit{if} \quad \deg(D) \geq N.$$

Proof. Choose E as above, and define k to be the smallest integer so that Lemma 21.5 holds for k. Suppose that D is a divisor on X such that $D \leq mE$ for some integer m. It follows from Exercise 21.2 that

$$\dim(L(D)) \;\geq\; \deg(D) + \dim(L(mE)) - \deg(mE) \;\geq\; \deg(D) + 1 - k,$$

which proves the required inequality for such a divisor D. Given any divisor D on X, there is a nonzero meromorphic function h such that $D - \mathrm{Div}(h) \leq mE$ for some integer m. Indeed, as in the preceding lemma, one can take h to be $\Pi(z - z(P))^{\mathrm{ord}_P(D)}$, the product over all P with $z(P) \neq \infty$. Then by Lemma 21.4(a) and the result just proved,

$$\begin{aligned} \dim(L(D)) &= \dim(L(D - \mathrm{Div}(h))) \\ &\geq \deg(D - \mathrm{Div}(h)) + 1 - k \;=\; \deg(D) + 1 - k. \end{aligned}$$

By the minimality of k, there is some divisor D_0 such that the dimension of $L(D_0)$ is $\deg(D_0) + 1 - k$. From Exercise 21.2 it follows that for any divisor D such that $D \geq D_0$,

$$\dim(L(D)) \;\leq\; \deg(D) + \dim(L(D_0)) - \deg(D_0) \;\leq\; \deg(D) + 1 - k,$$

so $\dim(L(D)) = \deg(D) + 1 - k$ for any such D.

Let $N = \deg(D_0) + k$. If the degree of D is at least N, then the degree of $D - D_0$ is at least k, so the dimension of $L(D - D_0)$ is at least $k + 1 - k = 1$. There is therefore a nonzero function f in $L(D - D_0)$, which means that $D + \mathrm{Div}(f) \geq D_0$, and so

$$\begin{aligned} \dim(L(D)) &= \dim(L(D + \mathrm{Div}(f))) \\ &= \deg(D + \mathrm{Div}(f)) + 1 - k \;=\; \deg(D) + 1 - k, \end{aligned}$$

which proves the lemma. $\square$

There can be only one integer k with the second property in the lemma, so k depends only on the Riemann surface X. We will see in §21c that k is the genus of X, and that N can be taken to be $2g-2$.

21b. Adeles

Lemma 21.1(b) says that each of the two subspaces $L(D) \subset L(D+Q)$ and $\Omega(D+Q) \subset \Omega(D)$ are either equalities or subspaces of codimension one. These cannot both be subspaces of codimension one, for if ω is in $\Omega(D)$ and f is in $L(D+Q)$, then $f \cdot \omega$ is a meromorphic 1-form with at most one simple pole at Q; the Residue Formula then implies that $\mathrm{Res}_Q(f \cdot \omega) = 0$, which means that $f \cdot \omega$ does not have a pole at Q, and hence either ω is in $\Omega(D+Q)$ or f is in $L(D)$. We will eventually see that one of these inclusions is an equality exactly when the other is not, and this is the core of the proof of the Riemann–Roch theorem. What we will do in this section is to prove a kind of local version of this assertion.

For a point P in X, let us denote by M_P the *germs* of meromorphic functions at P. These germs are defined as in §16b, by taking equivalence classes of meromorphic functions in neighborhoods of P, two being equivalent if they agree on some (punctured) neighborhood of P. If z is a local coordinate at P, any such germ has a unique power series expansion $\sum_{n=-m}^{\infty} a_n z^n$. If f is in M_P, and ω is a meromorphic 1-form on X, the *residue* $\mathrm{Res}_P(f \cdot \omega)$ can be defined to be $1/2\pi i$ times the integral of $f \cdot \omega$ around a small counterclockwise circle around P. In local coordinates, $f \cdot \omega$ can be written $\sum b_n z^n\, dz$, and this residue is b_{-1}.

Define an *adele* on X to be the assignment of a germ f_P of a meromorphic function at P for every point P in X, with the property that $\mathrm{ord}_P(f_P) \geq 0$ (i.e., f_P is holomorphic at P) for all but finitely many P. We write $\mathbf{f} = (f_P)$ for the adele defined by such a collection of functions f_P. These adeles form a complex vector space, which we denote by R. Any meromorphic function f on X determines an adele, by assigning the germ of f at P to each P, so the field M of meromorphic functions is a subspace of R. An adele can be thought of as a kind of "discontinuous function" on X. Since there is no relation between the "values" f_P at different points of X, it is remarkable that they can be a useful tool.

If $\mathbf{f} = (f_P)$ is an adele, then $\mathrm{Res}_P(f_P \cdot \omega) = 0$ for all but finitely many P (those where f_P or ω has a pole). We can therefore add the residues

$\mathrm{Res}_P(f_P \cdot \omega)$ over all P in X, getting a complex number. In other words, ω defines a homomorphism

$$\varphi_\omega \colon R \to \mathbb{C}, \qquad \mathbf{f} = (f_P) \mapsto \sum_{P \in X} \mathrm{Res}_P(f_P \cdot \omega),$$

which is a linear map of complex vector spaces. If $D = \Sigma m_P P$ is a divisor such that ω is in $\Omega(D)$, and $\mathrm{ord}_P(f_P) \geq -m_P$ for all P, then $f_P \cdot \omega$ is holomorphic at P, so the residue is zero. Define $R(D) \subset R$ by the formula

$$R(D) = \{\mathbf{f} = (f_P) \in R \colon \mathrm{ord}_P(f_P) \geq -\mathrm{ord}_P(D) \text{ for all } P \in X\}.$$

This means that the homomorphism φ_ω vanishes on $R(D)$. In addition, the Residue Formula says that if these f_P all come from one meromorphic function f on X, then $\Sigma_{P \in X} \mathrm{Res}_P(f \cdot \omega) = 0$, so φ_ω also vanishes on the subspace M of R. It follows that φ_ω determines a homomorphism (still denoted φ_ω)

$$\varphi_\omega \colon R/(R(D) + M) \to \mathbb{C}.$$

Define $S(D)$ to be this complex vector space $R/(R(D) + M)$, and define $\Omega'(D)$ to be the dual space

$$\Omega'(D) = S(D)^* = \mathrm{Hom}_{\mathbb{C}}(R/(R(D) + M), \mathbb{C}).$$

Then φ_ω is an element of this space $\Omega'(D)$. What we have done is construct a natural homomorphism from $\Omega(D)$ to $\Omega'(D)$, taking ω to φ_ω. In the next section we will show that this homomorphism is an isomorphism.

Exercise 21.7. Show that the homomorphism from $\Omega(D)$ to $\Omega'(D)$ is injective, and that a meromorphic 1-form ω is in $\Omega(D)$ if and only if φ_ω vanishes on $R(D)$.

In this section we prove that $\Omega'(D)$ has some properties we would like $\Omega(D)$ to have. As in the preceding section, the main idea is to compare $\Omega'(D)$ and $\Omega'(D + Q)$ for Q a point in X.

Since $R(D)$ is contained in $R(D + Q)$, there is a canonical surjection from $R/(R(D) + M)$ onto $R/(R(D + Q) + M)$, i.e., from $S(D)$ onto $S(D + Q)$. The kernel is $(R(D + Q) + M)/(R(D) + M)$.

Lemma 21.8. *For any divisor D and point Q,*

$$\dim((R(D + Q) + M)/(R(D) + M)) \leq 1,$$

with equality if and only if $L(D) = L(D + Q)$.

Proof. Choose a germ g_Q at Q such that $\mathrm{ord}_Q(g_Q) = -\mathrm{ord}_Q(D) - 1$, and let $g_P = 0$ for $P \neq Q$. The adele $\mathbf{g} = (g_P)$ gives a generator of the quotient space in the lemma. Indeed if $f = (f_P)$ is any adele in $R(D+Q)$, then there is some scalar λ so that $f - \lambda g$ is in $R(D)$. This element will be nonzero exactly when there is no h in M with $g - h$ in $R(D)$, which says exactly that there is no h in $L(D+Q)$ that is not in $L(D)$. □

Let k be the integer from Lemma 21.6.

Lemma 21.9.

(*a*) *If D is a divisor such that $\dim(L(D)) = \deg(D) + 1 - k$, then $R(D) + M = R$, so $S(D) = 0$ and $\Omega'(D) = 0$.*

(*b*) *For any D the space $S(D)$ has finite dimension, so its dual space $\Omega'(D)$ has the same finite dimension.*

(*c*) *For any divisor D and point Q, $\Omega'(D+Q)$ is a subspace of $\Omega'(D)$, and*

$$\dim(\Omega'(D)/\Omega'(D+Q)) \leq 1,$$

with equality if and only if $L(D) = L(D+Q)$.

(*d*) *For any nonzero meromorphic function f on X,*

$$\dim(\Omega'(D + \mathrm{Div}(f))) = \dim(\Omega'(D)).$$

Proof. If $\dim(L(D)) = \deg(D) + 1 - k$, then for any point Q we know that $\dim(L(D+Q)) = \deg(D+Q) + 1 - k$ (see Exercise 21.2), i.e., $L(D+Q) \neq L(D)$. By the preceding lemma, this means that $R(D+Q) \subset R(D) + M$. Continuing to add points to $D+Q$, we see that $R(E) \subset R(D) + M$ for all divisors E such that $E \geq D$. But any element of R is in $R(E)$ for some such E, so $R = R(D) + M$, which proves (a).

For (b), take a sequence of surjections $S(D) \twoheadrightarrow S(D+Q_1) \twoheadrightarrow S(D+Q_1+Q_2) \twoheadrightarrow \ldots \twoheadrightarrow S(E)$, until E is large enough so (a) implies that $S(E)$ is zero. Lemma 21.8 implies that the kernel of each of these surjections is at most one dimensional, so by induction each $S(D)$ must be finite dimensional.

Dual to the exact sequence

$$0 \to (L(D+Q)+M)/(L(D)+M) \to S(D) \to S(D+Q) \to 0$$

is the exact sequence

$$0 \to \Omega'(D+Q) \to \Omega'(D) \to ((L(D+Q)+M)/(L(D)+M))^* \to 0.$$

This shows that the inclusion $\Omega'(D+Q) \hookrightarrow \Omega'(D)$ is either an isomorphism or its cokernel has dimension one. By the preceding lemma, we see that the latter occurs exactly when $L(D) = L(D+Q)$, which proves (c).

For (d), there is a natural isomorphism from $R(D + \mathrm{Div}(f))$ to $R(D)$ that takes $\mathbf{f}$ to $f \cdot \mathbf{f}$. This determines an isomorphism from $S(D + \mathrm{Div}(f))$ to $S(D)$, and, taking duals, from $\Omega'(D)$ to $\Omega'(D + \mathrm{Div}(f))$. □

Lemma 21.10. *For any divisor D on X,*

$$\dim(L(D)) \;=\; \deg(D) + 1 - k + \dim(\Omega'(D)).$$

Proof. This equation is certainly true if $\deg(D) \geq N$, with N as in Lemma 21.6, for then $\dim(L(D)) = \deg(D) + 1 - k$ and $\dim(\Omega'(D)) = 0$ by Lemma 21.6 and Lemma 21.9(a). Since we can get between any two divisors by successively adding and subtracting points, it suffices to show that the equation is true for a divisor D if and only if it is true for $D + Q$, where Q is any point. Comparing the two equations, what must be proved is that

$$\dim(L(D+Q)) - \dim(L(D)) + \dim(\Omega'(D)) - \dim(\Omega'(D+Q)) \;=\; 1,$$

and this is simply a translation of (c) in the preceding lemma. □

Let Ω' be the union of all $\Omega'(D)$, taken over all divisors D. An element of Ω' is a homomorphism from R to $\mathbb{C}$ which vanishes on M and vanishes on some (unspecified) $R(D)$. The space Ω of meromorphic differentials on X maps to Ω', and we want to see that this is an isomorphism. We have seen that if ω is any nonzero meromorphic differential, any other can be written in the form $f \cdot \omega$ for some meromorphic function f. This means that Ω is a one-dimensional vector space over the field M. The space Ω' is also a vector space over M, by the rule that if f is in M and $\varphi\colon R \to \mathbb{C}$, then $f \cdot \varphi$ is the homomorphism which takes $\mathbf{f}$ to $\varphi(f \cdot \mathbf{f})$.

Lemma 21.11. *The dimension of Ω' over M is* 1.

Proof. We know that Ω' is not zero, for example, by applying the preceding lemma for D of small degree to see that $\Omega'(D) \neq 0$. To complete the proof we must show that two elements φ and ψ in Ω' cannot be independent over M. If this were the case, then for any elements $h_1, \ldots, h_n$ of M which are independent over $\mathbb{C}$, it would follow that $h_1\varphi, \ldots, h_n\varphi, h_1\psi, \ldots, h_n\psi$ are elements of Ω' which are independent over $\mathbb{C}$. Take a divisor E which is large enough so that φ and ψ are in $\Omega'(E)$. Suppose the functions h_i are a basis for $L(D)$ for some D. Then the $2n$ products $h_i\varphi$ and $h_i\psi$ are in $\Omega'(E - D)$, so

$$\dim(\Omega'(E-D)) \geq 2\dim(L(D)) \geq 2(\deg(D)+1-k).$$

By Lemma 21.10,

$$\dim(\Omega'(E-D)) = k-1-\deg(E-D)+\dim(L(E-D)).$$

Now $\deg(E-D) = \deg(E)-\deg(D)$, and $L(E-D)=0$ provided $\deg(E-D)<0$. So if we take any D with $\deg(D)>\deg(E)$, the displays lead to the inequality

$$2(\deg(D)+1-k) \leq \deg(D)-\deg(E)+k-1,$$

which says that $\deg(D) \leq 3k-3-\deg(E)$. But we may take D of arbitrarily large degree, which is a contradiction. □

The canonical homomorphism from the space Ω of meromorphic differentials to the space Ω' is a homomorphism of vector spaces over the field M. It is not identically zero by Exercise 21.7, since one can certainly find meromorphic differentials ω and adeles $\mathbf{f}$ such that there is exactly one point P at which $f_P \cdot \omega$ has a simple pole. But since both vector spaces have dimension one over M, it follows that the map $\Omega \rightarrow \Omega'$ is an isomorphism. We saw at the beginning that the subspace $\Omega(D)$ of Ω is mapped into $\Omega'(D)$ by this map, and it follows from Exercise 21.7 that $\Omega(D)$ maps isomorphically onto $\Omega'(D)$. So we have proved

Proposition 21.12. *The canonical map $\Omega(D) \rightarrow \Omega'(D)$ is an isomorphism.*

Exercise 21.13. Given germs $f_1, \ldots, f_n$ of meromorphic functions at distinct points $P_1, \ldots, P_n$ of X, and integers $m_1, \ldots, m_n$, show that there is a meromorphic function f so that $\operatorname{ord}_{P_i}(f-f_i) \geq m_i$ for $1 \leq i \leq n$.

21c. Riemann–Roch

From Lemma 21.10 and Proposition 21.12 we have the formula

$$(*) \qquad \dim(L(D)) = \deg(D)+1-k+\dim(\Omega(D))$$

for all divisors D on X, valid for some integer k which we do not yet know. We can specialize (*) to some cases where we know something. For example, if we take $D = 0$, $L(0) = \mathbb{C}$, and the formula says that $1 = 0 + 1 - k + \dim(\Omega(0))$, i.e., that the space of holomorphic 1-forms has dimension k. Fix any meromorphic divisor ω, and let K be the divisor of ω, which we know is a divisor of degree $2g - 2$. Applying Lemma 21.4(b) with $D = K$, we get

$$\dim(\Omega(K)) = \dim(L(K - K)) = \dim(L(0)) = 1,$$

and applying the same lemma with $D = 0$, we get

$$\dim(L(K)) = \dim(\Omega(0)) = k.$$

Now apply (*) with $D = K$, yielding

$$k = (2g - 2) + 1 - k + 1,$$

which means that k must be g. So we have proved:

Theorem 21.14 (Riemann–Roch Theorem). *If X is the Riemann surface of an algebraic curve, then for any divisor D on X,*

$$\begin{aligned}\dim(L(D)) &= \deg(D) + 1 - g + \dim(\Omega(D)) \\ &= \deg(D) + 1 - g + \dim(L(K - D)),\end{aligned}$$

where K is the divisor of any nonzero meromorphic 1-form on X.

Corollary 21.15. *The space of holomorphic differentials has dimension g.*

This proves that $H^1(X; \mathbb{C}) = \Omega^{1,0}(X) \oplus \Omega^{0,1}(X)$.

Corollary 21.16. *$\dim(L(D)) \geq \deg(D) + 1 - g$, with equality whenever $\deg(D) \geq 2g - 1$.*

Corollary 21.17. *For any two points P and Q on X, there is a meromorphic 1-form φ with simple poles at P and Q and no other poles.*

Proof. Riemann–Roch for $D = -P - Q$ gives

$$0 = -2 + 1 - g + \dim(\Omega(-P - Q)),$$

or $\dim(\Omega(-P - Q)) = g + 1$. This means that there is a meromorphic 1-form φ that is in $\Omega(-P - Q)$ but not in $\Omega(0)$. Since the sum of the residues is zero, φ must have simple poles at both P and Q, with the residue at Q being minus the residue at P. □

Corollary 21.18. *If $g_X = 0$, then X is isomorphic to S^2.*

Proof. Take any point P. Since $\dim(L(P)) \geq \deg(P) + 1 - g = 2$, there is a nonconstant meromorphic function f with at most one pole. This is a mapping from X to S^2 of degree 1, so is an isomorphism. □

Exercise 21.19. If $g_X = 1$, show that there is an analytic mapping $f\colon X \to S^2$ of degree 2. Deduce that X is the Riemann surface of a curve $W^2 = Z(Z-1)(Z-\lambda)$, $\lambda \neq 0, 1$.

Exercise 21.20. Assume that $g = g_X \geq 1$. (a) Show that there are g distinct points $P_1, \ldots, P_g$ on X so that $\Omega(P_1 + \ldots + P_g) = 0$. (b) Show that there are points $P_1, \ldots, P_g$ on X so that $\Omega(P_1 + \ldots + P_g) \neq 0$. (c) If $g \geq 2$, show that there is an analytic mapping $f\colon X \to S^2$ of degree at most g. In particular, if $g = 2$, X is hyperelliptic.

Exercise 21.21. Show that the 1-form φ of Corollary 21.17 is unique up to multiplying by a nonzero scalar and adding a holomorphic 1-form. Show that there is a unique such φ whose residue at P is 1, whose residue at Q is -1, and so that $\int_\gamma \varphi$ is purely imaginative for all closed paths γ in $X \setminus \{P, Q\}$.

Exercise 21.22. A real-valued function u on a Riemann surface is *harmonic* if it is locally the real part of an analytic function. A function which is harmonic in a punctured neighborhood of a point P is said to have a *logarithmic pole* at P if, with z a local coordinate at P, there is a nonzero real scalar a so that $u - a \cdot \log(|z|)$ extends to be harmonic in a neighborhood of P. Show that for any two points P and Q on X, there is a harmonic function u on $X \setminus \{P, Q\}$ that has logarithmic poles at P and Q.

Exercise 21.23. For any point P on X, show that there is a meromorphic differential φ on X with a double pole at P. Deduce that there is a harmonic function u on $X \setminus \{P\}$ so that, if $z = x + iy$ is a local coordinate at P, then $u - x/(x^2 + y^2)$ is harmonic near P.

Historically, the arguments went in the reverse order: harmonic functions were found with the properties in the preceding exercises, and these were used to find meromorphic 1-forms and to prove Riemann–Roch. By regarding harmonic functions as integrals of fluid flows or electric fields on X, one can give intuitive arguments for their existence, say by putting sources and sinks at the points P and Q. For a lively discussion along these lines, see Klein (1893).

Exercise 21.24. Given any sequence $P_1, P_2, \ldots, P_n, \ldots$ of points in X, show that there are exactly g positive integers k, all in the interval $[1, 2g-1]$, such that

$$L(P_1 + \ldots + P_{k-1}) = L(P_1 + \ldots + P_k).$$

When all P_i are taken to be a fixed point P, these integers are called the *Weierstrass gaps at P*.

Exercise 21.25. Suppose D and E are divisors on X such that $D + E$ is the divisor of a meromorphic 1-form. Prove *Brill–Noether reciprocity:* $\dim(L(D)) - \dim(L(E)) = {}^1\!/_2(\deg(D) - \deg(E))$.

Exercise 21.26. If $z\colon X \to S^2$ is an analytic mapping of degree n, and Q in S^2 is a point such that $z^{-1}(Q) = \{P_1, \ldots, P_n\}$ has n distinct points, use Riemann–Roch to show that there is a meromorphic function w on X so that $w(P_1), \ldots, w(P_n)$ are distinct complex numbers. If $F(z, w) = 0$ is the irreducible equation for w over $\mathbb{C}(z)$ (with denominators cleared), show that X is isomorphic to the Riemann surface of F.

Once one knows Riemann–Roch for a general compact Riemann surface X, the preceding exercise shows that X comes from an algebraic curve.

Exercise 21.27. Show that the Riemann surface X of the polynomial $W^4 + Z^4 - 1 = 0$ has genus 3, but X is not hyperelliptic. Show in fact that $\dim\Omega(2P) = 1$ for all P in X.

21d. The Abel–Jacobi Theorem

In this section we prove the assertions made in §20d. The first is Abel's criterion for when a divisor $D = \Sigma m_i P_i$ is a divisor of a meromorphic function: it must have degree zero and be in the kernel of the Abel–Jacobi map. We use the notation of that section.

Theorem 21.28 (Abel's Theorem). *There is a meromorphic function f on X with* $\mathrm{Div}(f) = D$ *if and only if* $\deg(D) = 0$ *and* $[D] = 0$ *in* $J(X)$.

We first sketch the proof of the necessity of these conditions. Suppose f is a nonconstant meromorphic function on X, giving a mapping of degree n from X to S^2, with branch set S. Corollary 19.5 shows that $\deg(\mathrm{Div}(f)) = 0$. Fix a point P_0 in X. Consider the mapping from $S^2 \setminus S$ to $J(X)$ that takes a point Q to the point $\Sigma_{i=1}^{n} [P_i - P_0]$, where $P_1 \ldots , P_n$ are the points of X in $f^{-1}(Q)$. It is not hard to see that this extends continuously to the branch points, giving a continuous mapping from S^2 to $J(X)$. Since S^2 is simply connected, by Proposition 13.5 this mapping factors: $S^2 \to \mathbb{C}^g \to \mathbb{C}^g/\Lambda$. By looking locally, one can verify that each of the g coordinate maps are analytic functions on S^2. But analytic functions on S^2 are constant, so the given map from S^2 to $J(X)$ must be constant. The fact that the value at 0 is equal to the value at ∞ is precisely the condition that $[\mathrm{Div}(f)] = [\Sigma \operatorname{ord}_P(f) \cdot P] = 0$ in $J(X)$.

We turn now to the converse. Let D be a divisor of degree zero in the kernel of the Abel–Jacobi map. We must show that D is the divisor of a meromorphic function. The motivation for the proof comes from the fact that if f is a meromorphic function on X, and we set $\varphi = df/f$, then φ is a meromorphic differential on X with at most simple poles and, in fact, for any P, $\mathrm{Res}_P(\varphi) = \operatorname{ord}_P(f)$. We will look for a meromorphic 1-form φ with at most simple poles among the points appearing in D, such that $\mathrm{Res}_P(\varphi) = \operatorname{ord}_P(D)$ for all P. Then we will define a function f on X by the formula $f(P) = \exp(\int_{P_0}^{P} \varphi)$, where P_0 is a fixed point. Provided this is well defined, it will satisfy the equation $\varphi = df/f$, and so we will have $\operatorname{ord}_P(f) = \operatorname{ord}_P(D)$ for all P, so $\mathrm{Div}(f) = D$.

Since the degree of D is zero, we may write $D = \Sigma_{i=1}^{r}(P_i - Q_i)$, for some points $P_1, \ldots , Q_r$ (not necessarily unique). Let $S = \{P_1, \ldots , P_r, Q_1, \ldots , Q_r\}$. We know by Corollary 21.17 that there is a meromorphic 1-form φ_i with simple poles at P_i and Q_i (only), and with residues 1 at P_i and -1 at Q_i. Let $\varphi = \Sigma_{i=1}^{r} \varphi_i$. We want to define $f(P) = \exp(\int_{P_0}^{P} \varphi)$, where the integral is along any path from P_0 to P in $X \setminus S$. This will be well defined provided the integral of φ along any closed path τ in $X \setminus S$ is in $2\pi i\mathbb{Z}$. The form φ is only defined up to the addition of a holomorphic 1-form, so the proof of the Abel–Jacobi theorem is reduced to the

Claim 21.29. *There is a holomorphic* 1*-form* ω *so that* $\int_\tau(\varphi - \omega)$ *is in* $2\pi i\mathbb{Z}$ *for all* 1*-cycles* τ *on* $X \setminus S$.

We need the following refinement of Exercise 18.8. We take $2g$ closed arcs a_j and b_j as in §17c. Cutting the surface open along these arcs, we realize it as the identification space of a plane polygon Π with sides identified. These choices can be made so that the map from Π to X is a diffeomorphism on the interior of Π, and has a $\mathscr{C}^\infty$ extension to a neighborhood of Π. By means of this map 1-forms on X correspond to 1-forms on Π. Let ω be a closed 1-form on X, and define a function h on the closure of Π by the formula $h(P) = \int_{P_0}^{P} \omega$, for some fixed point P_0 in Π. Let φ be a closed 1-form defined on a neighborhood of the union of these $2g$ arcs in X, so φ determines a 1-form on a neighborhood of the boundary $\partial\Pi$ of Π.

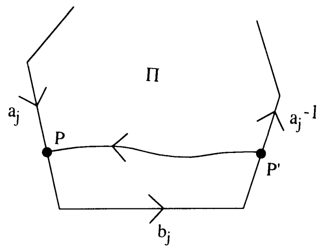
Lemma 21.30.

$$\int_{\partial\pi} h\varphi = \sum_{j=1}^{g} \left(\int_{a_j} \omega \int_{b_j} \varphi - \int_{a_j} \varphi \int_{b_j} \omega \right).$$

Proof. Note first that if P and P' are corresponding points of the boundary edges a_j and a_j^{-1} of Π, then

$$h(P) - h(P') = -\int_{b_j} \omega ,$$

as one sees by integrating along a path from P' to P, noting that the integrals over corresponding parts of a_j and a_j^{-1} cancel.



Therefore

$$\begin{aligned} \int_{a_j} h\varphi + \int_{a_j^{-1}} h\varphi &= \int_{a_j} (h(P) - h(P'))\, \varphi(P) \\ &= \int_{a_j} \left(-\int_{b_j} \omega \right) \varphi = -\int_{b_j} \omega \cdot \int_{a_j} \varphi . \end{aligned}$$

Similarly if Q and Q' are corresponding points of b_j and b_j^{-1}, then

$$h(Q) - h(Q') = \int_{a_j} \omega ,$$

so

$$\int_{b_j} h\varphi + \int_{b_j^{-1}} h\varphi = \int_{b_j} (h(Q) - h(Q'))\, \varphi(Q) = \int_{a_j} \omega \cdot \int_{b_j} \varphi .$$

Adding over all the edges of the boundary of Π gives the identity of the lemma. □

To apply the lemma in our situation, the arcs a_j, b_j must be taken so that none of them goes through a point of S. Let σ_i be a path from Q_i to P_i which does not hit any of the arcs a_j or b_j.

Lemma 21.31. *For any holomorphic* 1*-form* ω *on* X,

$$(2\pi i)\sum_{i=1}^{r}\int_{\sigma_i}\omega = \sum_{j=1}^{g}\left(\int_{a_j}\omega\int_{b_j}\varphi - \int_{a_j}\varphi\int_{b_j}\omega\right).$$

Proof. We apply the preceding lemma. We must evaluate $\int_{\partial\Pi} h\varphi$, with $h(P) = \int_{P_0}^{P}\omega$. The Residue Formula in the polygon Π gives

$$\int_{\partial\Pi} h\varphi = (2\pi i)\sum_{p\in\Pi}\mathrm{Res}_P(h\varphi) = (2\pi i)\left(\sum_{i=1}^{r}h(P_i) - \sum_{i=1}^{r}h(Q_i)\right),$$

the last since φ_i has simple poles, with residues 1 and -1 at P_i and Q_i. And

$$\sum_{i=1}^{r}h(P_i) - \sum_{i=1}^{r}h(Q_i) = \sum_{i=1}^{r}\int_{\sigma_i}\omega$$

by definition. So Lemma 21.30 gives the required conclusion. □

The hypothesis that D is in the kernel of the Abel–Jacobi map means that there is a 1-cycle γ such that $\Sigma_{i=1}^{r}\int_{\sigma_i}\omega = \int_\gamma\omega$ for all holomorphic 1-forms ω. So we have, for all holomorphic ω,

$$(2\pi i)\int_\gamma\omega = \sum_{j=1}^{g}\left(\int_{a_j}\omega\int_{b_j}\varphi - \int_{a_j}\varphi\int_{b_j}\omega\right).$$

We can now prove Claim 21.29. Since a basis of $H_1(X\setminus S)$ is given by the cycles a_j, b_j, and small circles around the points in S (either by Mayer–Vietoris or Problem 17.12), it suffices to find a holomorphic 1-form ω such that the integral of $\varphi-\omega$ around all these cycles is in $2\pi i\mathbb{Z}$. Note that for any such φ, the integral around a small circle around a point in S is in $2\pi i\mathbb{Z}$, since all the residues of φ are integers. To start, let $\lambda_j = \int_{a_j}\varphi$. Subtracting the holomorphic 1-form $\Sigma\lambda_j\omega_j$ from φ, we can assume that $\int_{a_j}\varphi = 0$ for all j. The preceding formula, with $\omega = \omega_k$, gives

$$(2\pi i)\int_\gamma\omega_k = \int_{b_k}\varphi.$$

Write $\gamma = \Sigma m_j a_j + n_j b_j$, with integer coefficients m_j, n_j. Then

$$\begin{aligned}\int_\gamma \omega_k &= \sum_{j=1}^{g}\left(m_j\int_{a_j}\omega_k + n_j\int_{b_j}\omega_k\right)\\ &= m_k + \sum_{j=1}^{g} n_j\int_{b_j}\omega_k = m_k + \sum_{j=1}^{g} n_j\int_{b_k}\omega_j.\end{aligned}$$

the last step by the symmetry of Corollary 20.22. But now if we set $\omega = 2\pi i \Sigma n_j \omega_j$, this shows that $\int_{b_k}(\varphi - \omega) = 2\pi i m_k$. Therefore

$$\int_{a_k}(\varphi - \omega) = -\int_{a_k}\omega = -2\pi i\int_{a_k}\sum n_j\omega_j = -2\pi i n_k,$$

and this completes the proof of the claim, and hence of Abel's theorem.

Theorem 21.32 (Jacobi Inversion). *The Abel–Jacobi map from the group of cycles of degree zero to the Jacobian $J(X)$ is surjective.*

This means that we have an exact sequence

$$0 \to \mathbb{C}^* \to M(X)^* \to \widetilde{Z}_0(X) \to J(X) \to 0,$$

which realizes the torus $J(X) = \mathbb{C}^g/\Lambda$ as the quotient of the group of divisors of degree zero by the subgroup of divisors of meromorphic functions.

The proof, which we only sketch, requires a few basic facts about holomorphic mappings of several complex variables. Let X^g be the g-fold Cartesian product of X with itself, which is a g-dimensional complex manifold. Let

$$\alpha_g \colon X^g \to J(X)$$

be the map which takes $(P_1, \ldots, P_g)$ to $A(\Sigma P_i - gP_0)$, where A is the Abel–Jacobi map, and P_0 is any fixed point on X. It suffices to prove that α_g is surjective. The Jacobian $J(X) = \mathbb{C}^g/\Lambda$ gets the structure of a complex manifold so that the quotient mapping $\mathbb{C}^g \to J(X)$ is a local isomorphism.

Exercise 21.33. Verify that α_g is a holomorphic (analytic) mapping of complex manifolds.

We claim next that there are distinct points $P_1, \ldots, P_g$ in X such that the Jacobian determinant of α_g at the point $(P_1, \ldots, P_g)$ is not zero. Once this is verified, it follows that the image of α_g contains an open set. Since the image of the Abel–Jacobi map is a subgroup, and $J(X)$ is compact, it follows that the image must be all of $J(X)$. To prove the claim, take any distinct points $P_1, \ldots, P_g$. Let z_j be a local coordinate at P_j, and write $\omega_i = f_{i,j}\, dz_j$ near P_j.

Exercise 21.34. Verify that, in suitable coordinates, the Jacobian matrix of α_g at $(P_1, \ldots, P_g)$ is $(f_{i,j}(P_j))$.

Now take the points $P_1, \ldots, P_g$ as in Exercise 21.20(a). If the Jacobian determinant $\det(f_{i,j}(P_j))$ of α_g vanishes at $(P_1, \ldots, P_g)$, then there are g complex numbers $\lambda_1, \ldots, \lambda_g$, not all zero, such that

$$(\lambda_1\omega_1 + \ldots + \lambda_g\omega_g)(P_j) = 0$$

for all j. But this means that $\Omega(P_1 + \ldots + P_g)$ is not zero, which contradicts Exercise 21.20(a).

PART XI

HIGHER DIMENSIONS

This last part is designed to introduce the reader to a few of the higher-dimensional generalizations of the ideas we have studied in earlier chapters, both to unify these ideas, and to indicate a few of the directions one may go if one continues in algebraic topology. It is not written as the culmination or goal of the rest of the course, but rather as a brief introduction to the general theory. How accessible or useful it may be depends on several factors, such as background in manifold theory, and ability to generalize from the special cases we have seen to higher dimensions (an ability, it seems to me, often underestimated in our teaching). For systematic developments of the ideas of this part, the books of Bredon (1993), Bott and Tu (1980), Greenberg and Harper (1981), and Massey (1991) are recommended.

We have studied the first homology group H_1X, the fundamental group $\pi_1(X,x)$, and the first De Rham cohomology group H^1X, which were sufficient to capture most of the topology of the spaces we have been most concerned with: open sets in the plane and surfaces, and an occasional graph. Each of these is the first in a sequence of groups that are used to study similar questions for higher-dimensional spaces. In contrast to many earlier chapters, the tone in this final part is designed to be more formal, concise, and abstract; we are depending on your experience with special cases and low-dimensional examples for motivation.

We start by recalling some three-dimensional calculus, to indicate the sort of "topology" these higher groups might measure. Then we take a quick look at knots in 3-space, mainly because knot theory is an interesting and important subject in its own right, and also because it gives us a chance to use some of the tools developed in earlier chapters. We also define the higher homotopy groups of a space and the De Rham cohomology groups.

In the next chapter we define higher homology groups, and prove their basic properties. We indicate some of the ways they can be used to extend ideas we have looked at in the plane or on surfaces to higher dimensions. In particular, they give a simple extension of the notion of degree, and they lead to generalizations of the Jordan curve theorem.

In the final chapter we include a couple of "diagram chasing" facts from algebra, one of which can be used to compare different homology and cohomology groups, the other to construct long exact sequences such as Mayer–Vietoris sequences. (Having proved this once and for all, one does not have to keep doing the same sort of manipulations we have done to define boundary and coboundary maps.) Finally, we give proofs of the basic duality theorems between homology and De Rham cohomology on manifolds.

The sections involving higher De Rham cohomology on manifolds are written for those with a working knowledge of differential forms on differentiable manifolds. A reader without this background can skip or skim this part (or stick to low dimensions and/or open sets in $\mathbb{R}^n$). The construction of higher homology groups and its applications to higher-dimensional analogues of the theorems we saw at the beginning of the course do not depend on any of this, however, and a reader who has mastered the earlier chapters should be able to work through this without any gaps. We have included Borsuk's theorem that maps of spheres that preserve antipodal pairs have odd degree, since this allows generalizing all the results we proved about winding numbers to higher dimensions.

A few remarks about other approaches may be in order. It is possible to define the degree of a mapping of a sphere to itself without the notions of homology, and to prove many of its properties. Those with a knowledge of differential topology can do this by approximating a continuous map by a differentiable one, and following the pattern of Problem 3.32 for the winding number. (For a nice discussion in the $\mathscr{C}^\infty$ context, see Milnor (1965)). There are also elementary approaches using simplicial approximations, although considerable care is required to make the arguments rigorous. (In fact, the difficulties in this approach seem to us much greater than those involved in developing general homology theory—not to mention the fact that, having done the latter, one can apply it in many other situations.)

We might also mention that we have used cubes rather than the simplices that are common in many other treatments. Simplices have a slight advantage that one has no "degenerate" maps to ignore, but cubes are simpler for homotopies and product spaces in general, and they are more convenient for integrating differential forms. (In fact, it is not hard—as we do in §24d—to use cubes to calculate the simplicial homology of spaces that are triangulated.)

Finally, a word on terminology: here *manifolds* always are assumed to have a countable basis for their topology.

CHAPTER 22

Toward Higher Dimensions

22a. Holes and Forms in 3-Space

On an open set U in $\mathbb{R}^3$ we have: *0-forms*, which are just $\mathscr{C}^\infty$ functions on U; *1-forms*, which are expressions

$$p\,dx + q\,dy + r\,dz,$$

where p and q and r are $\mathscr{C}^\infty$ functions on U; *2-forms*, which are expressions

$$u\,dy\,dz + v\,dx\,dz + w\,dx\,dy,$$

where u and v and w are $\mathscr{C}^\infty$ functions on U; and *3-forms*, which are expressions

$$h\,dx\,dy\,dz,$$

where h is a $\mathscr{C}^\infty$ function on U.

These forms are designed for integrating, just as in the plane. A 0-form is evaluated at points. The integral of a 1-form over a differentiable path $\gamma\colon [a,b] \to U$ is defined exactly as for the plane:

$$\int_\gamma p\,dx + q\,dy + r\,dz \;=\; \int_a^b \left(p(\gamma(t))\frac{dx}{dt} + q(\gamma(t))\frac{dy}{dt} + r(\gamma(t))\frac{dz}{dt}\right) dt,$$

where $\gamma(t) = (x(t), y(t), z(t))$. A 2-form can be integrated over a differentiable map $\Gamma\colon R \to U$ from a rectangle $R = [a,b]\times[c,d]$ to U:

$$\iint_\Gamma u\,dy\,dz + v\,dx\,dz + w\,dx\,dy \;=$$

$$\iint_R \left(u(\Gamma(s,t))\frac{\partial(y,z)}{\partial(s,t)} + v(\Gamma(s,t))\frac{\partial(x,z)}{\partial(s,t)} + w(\Gamma(s,t))\frac{\partial(x,y)}{\partial(s,t)}\right) ds\,dt,$$

where, for $\Gamma(s,t) = (x(s,t), y(s,t), z(s,t))$, and $\partial(x,y)/\partial(s,t)$ denotes $\partial x/\partial s\ \partial y/\partial t - \partial x/\partial t\ \partial y/\partial s$, and similarly for the other terms. A 3-form is integrated over a differentiable map $\Pi\colon B \to U$ where B is a rectangular box $[a,b]\times[c,d]\times[e,f]$:

$$\iiint_\Pi h\,dx\,dy\,dz \;=\; \iiint_B h(\Pi(s,t,u))\frac{\partial(x,y,z)}{\partial(s,t,u)}\,ds\,dt\,du,$$

where $\Pi(s,t,u) = (x(s,t,u), y(s,t,u), z(s,t,u))$, and $\partial(x,y,z)/\partial(s,t,u)$ denotes the Jacobian determinant.

The *differential* df of a 0-form f is a 1-form defined by

$$df \;=\; \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

The fundamental theorem of calculus gives $\int_\gamma df = f(\gamma(b)) - f(\gamma(a))$ for γ a path as above. The differential of a 1-form is a 2-form:

$$d(p\,dx + q\,dy + r\,dz)$$
$$= \left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}\right) dy\,dz + \left(\frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}\right) dx\,dz + \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx\,dy,$$

and Green's theorem (for rectangles in the plane) gives $\iint_\Gamma d\omega = \int_{\partial\Gamma}\omega$, where the integral around the boundary of the rectangle is defined as in Part I. Finally, the differential of a 2-form is a 3-form:

$$d(u\,dy\,dz + v\,dx\,dz + w\,dx\,dy) \;=\; \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dx\,dy\,dz,$$

and Stokes theorem says that $\iiint_\Pi d\omega = \iint_{\partial\Pi}\omega$.

Exercise 22.1. State and prove Stokes' theorem for a box, and define the boundary of Π as a sum and difference of the restrictions of Π to the six sides of the box, assigning correct signs to each so the above formula holds.

We define the differential of a 3-form to be 0. A simple calculation using the equality of mixed derivatives shows that if f is a 0-form, then $d(df) = 0$, and if ω is a 1-form, then $d(d\omega) = 0$. A k-form ω is *closed* if $d\omega = 0$, and *exact* if $\omega = d\mu$ for some $(k-1)$-form μ. So all exact forms are closed, and we have the same question as in the plane: when are closed forms exact? We can define the De Rham groups H^kU as before: for $k = 0$, 1, 2, or 3,

$$H^kU \;=\; \{\text{closed } k\text{-forms on } U\}/\{\text{exact } k\text{-forms on } U\}.$$

The question becomes: How does the topology of U influence the size of the vector spaces H^kU?

For 1-forms, the answer is very similar to the case of open sets in the plane. A closed 1-form ω on U is exact if and only if integral of ω over paths in U depend only on the endpoints, or all integrals over closed paths are zero. For example, if U is the complement of the z-axis, the 1-form $\omega = (-y\,dx + x\,dy)/(x^2 + y^2)$ is closed but not exact; as we know, an integral of ω around a circle in the xy-plane is 2π. Notice that taking a point or a closed ball out of $\mathbb{R}^3$ does not count as a "hole" as far as 1-forms in 3-space is concerned. In fact:

Exercise 22.2. Show that if $H_1U = 0$ then every closed 1-form on U is exact.

For 2-forms, however, if U is the complement of a point, there are closed forms that are not exact. For example, let

$$\omega \;=\; \frac{x\,dy\,dz - y\,dx\,dz + z\,dx\,dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

Exercise 22.3. Show that ω is closed. Fix a positive number ρ, and let Γ: $[0, 2\pi] \times [-{}^1\!/\!{}_2\pi, {}^1\!/\!{}_2\pi] \to \mathbb{R}^3$ be the spherical coordinate mapping, i.e.,

$$\Gamma(\vartheta, \varphi) \;=\; (\rho\cos(\vartheta)\cos(\varphi), \rho\sin(\vartheta)\cos(\varphi), \rho\sin(\varphi)).$$

Compute the integral of ω over Γ, and deduce that ω is not exact.

Exercise 22.4. Use this 2-form ω to define the *engulfing number* around 0 of a differentiable map from S^2 to $\mathbb{R}^3 \setminus \{0\}$. Can you prove any analogues of the winding number?

We will see in the next chapter how to define second homology groups H_2U that have the same relation to 2-forms and H^2U as the first homology H_1U has to 1-forms and H^1U.

Some of these ideas may be more familiar in vector field language:

a 1-form $p\,dx + q\,dy + r\,dz$ can be identified with the vector field $p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$, the 2-form $u\,dy\,dz + v\,dx\,dz + w\,dx\,dy$ with the vector field $u\mathbf{i} - v\mathbf{j} + w\mathbf{k}$, and the 3-form $h\,dx\,dy\,dz$ with the function h. In this language, the differential df of a function corresponds to the *gradient* grad(f), the differential of a 1-form to the *curl* of a vector field, and the differential of a 2-form becomes the *divergence* of a vector field:

$$\begin{aligned}
\operatorname{grad}(f) &= \frac{\partial x}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k};\\
\operatorname{curl}(p\mathbf{i} + q\mathbf{j} + r\mathbf{k}) &= \left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial p}{\partial z} - \frac{\partial r}{\partial x}\right)\mathbf{j} + \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)\mathbf{k};\\
\operatorname{div}(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z};
\end{aligned}$$

and the equations $d \circ d = 0$ say that $\operatorname{curl} \circ \operatorname{grad} = 0$ and $\operatorname{div} \circ \operatorname{curl} = 0$. The integral of the vector field corresponding to a 2-form over a surface can be interpreted as the integral of a dot product with an outward-pointing normal.

Problem 22.5. (a) Generalize the discussion of §2c to fluid flows on open sets in 3-space. Interpret the engulfing number as the flux across a surface of a flow with source at the origin (see Exercise 2.26). (b) Define harmonic functions of three variables, and generalize Exercise 2.21 and Problems 2.22–2.25.

22b. Knots

A *knot* is a subset K of $\mathbb{R}^3$ or the 3-sphere S^3 that is homeomorphic to a circle. Call two knots *equivalent* if there is a homeomorphism of $\mathbb{R}^3$ (or S^3) with itself that takes one homeomorphically onto the other, and is orientation-preserving. (For a precise definition of "orientation-preserving," see §23c.) A weaker notion is *similarity*, which is the same except ignoring orientation, so "mirror image" knots are always similar. The generalized Jordan curve theorem, which we will prove in the next chapter, implies that the complement of X is connected, that its first homology group is infinite cyclic, and its other homology groups vanish. In particular, these groups are the same for all knots. However, the fundamental group $\pi_1(S^3 \setminus X)$ or $\pi_1(\mathbb{R}^3 \setminus X)$ can sometimes be used to distinguish knots from each other. Note that these fundamental groups are the same for similar knots, so they can be used as a possible invariant.

Exercise 22.6. If $K \subset \mathbb{R}^3$ is a knot, and $\mathbb{R}^3$ is identified with the complement of a point in S^3, by stereographic projection, show that the fundamental group of $\mathbb{R}^3 \setminus K$ is isomorphic to the fundamental group of $S^3 \setminus K$.

The standard embedding $K = S^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset S^3$ is called the trivial knot, and any knot equivalent to this is called *trivial*. We want to use fundamental groups to give one example of a knot that is not trivial. We will do this by identifying $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$ with the set of pairs (z, w) of complex numbers such that $|z|^2 + |w|^2 = 1$, and constructing a knot K that is the intersection of S^3 with a complex plane curve that has a singularity at the origin. This is more than a convenient way to find an example: the knots that arise this way are important invariants of singularities of plane curves! In this language, the trivial knot can be realized as the intersection of S^3, with the "curve" $w = 0$.

Exercise 22.7. Show that the fundamental group of the complement of the trivial knot is isomorphic to $\mathbb{Z}$. Show, in fact, that the circle $\{(0, w)\colon |w| = 1\}$ is a deformation retract of the complement of circle $\{(z, 0)\colon |z| = 1\}$ in S^3.

In nature coverings often arise by starting with a mapping that is not a covering, but becomes one after throwing away a locus where it fails to be a covering. We have seen this for an analytic mapping between Riemann surfaces in Chapter 19, where only a finite set had to be thrown away. With appropriate hypotheses, such a mapping is called a "branched covering," with the bad set the "branch locus." Here is another example. Consider the mapping $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ given by the formula $f(u, v) = (u, v^3 + uv)$. The inverse image of a point (z, w) has three points if the equation $v^3 + zv = w$ has three distinct solutions for the variable v, and one or two points otherwise.

Exercise 22.8. Show that $f^{-1}(z, w)$ has three points if and only if $4z^3 + 27w^2 \neq 0$. If $4z^3 + 27w^2 = 0$, but $(z, w) \neq (0, 0)$, the inverse image has two points, and for $(z, w) = (0, 0)$, the inverse image has one point.

Let $V \subset \mathbb{C}^2$ be the plane curve $4z^3 + 27w^2 = 0$, which is the branch locus of the above mapping f. Let K be the intersection of V with S^3:

$$V = \{(z, w)\colon 4z^3 + 27w^2 = 0\}, \qquad K = V \cap S^3.$$

We claim first that K is homeomorphic to a circle.

Exercise 22.9. Show that the mapping $e^{2\pi it} \mapsto (-ae^{4\pi it}, be^{6\pi it})$, where a is the positive solution to the equation $4a^3 + 27a^2 = 27$ and $b = \sqrt{1-a^2}$, is a homeomorphism of S^1 onto K.

We will consider the mapping $\mathbb{C}^2 \setminus f^{-1}(V) \to \mathbb{C}^2 \setminus V$ determined by f, and the restriction

$$p: Y = f^{-1}(S^3 \setminus K) \to S^3 \setminus K = X, \qquad (u, v) \mapsto (u, v^3 + uv).$$

Take $x = (1, 0)$ as the base point in X, and $y = (1, 0)$ as the base point in Y. Note that $p^{-1}(x) = \{(1,0), (1,i), (1,-i)\}$.

Claim 22.10. (1) *p is a three-sheeted covering map*; (2) *Y is connected*; *and* (3) *p is not a regular covering.*

It follows from this claim that $\pi_1(S^3 \setminus K, x)$ is not abelian, since every connected covering of a manifold with an abelian fundamental group is regular. In particular, K is not a trivial knot. We leave the proofs of (1)–(3) as exercises, with the following comments. The essential point of (1) is showing that the roots of a polynomial are, locally where the roots are distinct, continuous functions of the coefficients. In fact, they are analytic functions, by the same argument as in §20a. For (2), it suffices to show that the three points of $f^{-1}(x)$ can be connected by paths. Consider the loop $\gamma(t) = (e^{2\pi it}, 0)$, $0 \le t \le 1$, at x. This lifts to the loop $\tilde{\gamma}_1(t) = (e^{2\pi it}, 0)$ at y, and to the path $\tilde{\gamma}_2(t) = (e^{2\pi it}, ie^{\pi it})$ that goes from $(1, i)$ to $(1, -i)$ in Y.

Exercise 22.11. Find a path of the form $\sigma(t) = (\lambda(t), i\mu(t))$, $0 \le t \le 1$, with $\lambda(t) > 0$ and $\mu(t) \ge 0$ for all t, that goes from $(1, 0)$ to $(1, i)$ in Y.

This exercise shows that Y is connected, and since $\tilde{\gamma}_1$ is closed, and $\tilde{\gamma}_2$ is not, the covering is not regular.

In fact, up to equivalence, K is an example of a *torus knot*. The torus $T = \mathbb{R}^2/\mathbb{Z}^2$ sits in S^3 by the mapping that takes (x, y) to $((1/\sqrt{2})e^{2\pi ix}, (1/\sqrt{2})e^{2\pi iy})$. For any relatively prime pair of positive integers p and q, the image in the torus of the line with equation $qy = px$ in $\mathbb{R}^2$ is a knot that winds p times around the torus one way while it winds q times around the other way. This is called a torus knot of type (p, q).

Exercise 22.12. (a) Show that the knot $K = V \cap S^3$ considered above is equivalent to a torus knot of type $(2, 3)$. (b) Show that for relatively prime positive integers p and q the intersection of $z^q = w^p$ with S^3 is a torus knot of type (p, q).

The Van Kampen theorem can be used to calculate the fundamental group of the complement of any torus knot. The 3-sphere is the union of two solid tori

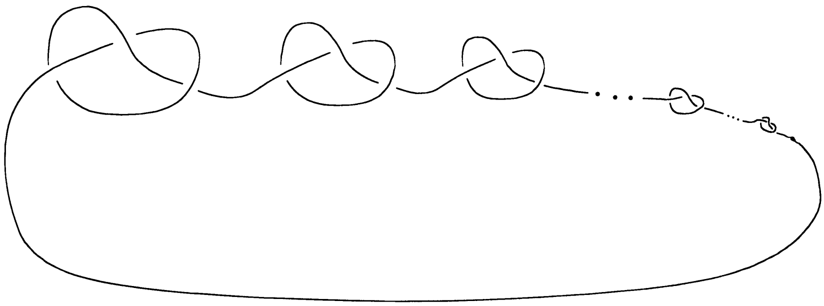
$$A = \{(z, w) \in S^3: |z| \leq |w|\}, \qquad B = \{(z, w) \in S^3: |z| \geq |w|\},$$

so $T = A \cap B = \{(z, w): |z| = |w| = 1/\sqrt{2}\}$ is a torus.

Problem 22.13. If K is a torus knot of type (p, q) in T, show that the fundamental groups of $A \setminus K$, $B \setminus K$, and $T \setminus K$ are infinite cyclic, and the generator of the fundamental group of $T \setminus K$ maps to the pth and qth powers of generators of the fundamental groups of $A \setminus K$ and $B \setminus K$. Apply the Van Kampen theorem to show that the fundamental group of $S^3 \setminus K$ has two generators a and b, and one relation $a^p \cdot b^q = e$, i.e., the fundamental group is F_2/N, where F_2 is the free group on a and b, and N is the least normal subgroup containing $a^p \cdot b^q$.

For a knot of type $(2, 3)$, for example, one can see again that this group is not abelian by mapping it onto the symmetric group $\mathfrak{S}_3$ on three letters, sending a to the transposition $(1\ 2)$ and b to the permutation $(1\ 2\ 3)$.

There are many knots that are not torus knots. For example, one can take a torus knot, and take a small tube around it, which is homeomorphic to another torus, and put a torus knot on this. Repeating this construction arbitrarily often gives a class of knots which, remarkably, are exactly the knots one gets from singularities of plane curves. There are many other knots, however. Moreover, there are some "wild" knots, such as "Antoine's necklace":



The fundamental group of the complement of this knot is not even finitely generated. A piece of this is an embedding of a closed interval in $\mathbb{R}^3$ such that the complement is not simply connected.

22c. Higher Homotopy Groups

The higher homotopy groups $\pi_k(X,x)$ are easier to define than higher homology or cohomology groups, although their calculation turns out to be far more challenging. Fix a base point s_0 in the sphere S^k, say the north pole: $s_0=(0, \ldots, 0, 1)$. Define $\pi_k(X,x)$ to be the set of homotopy classes of maps from S^k to X that map s_0 to x; here a homotopy between two such maps must preserve basepoints throughout the homotopy, i.e., H is a continuous map from $S^k \times [0,1]$ to X, with $H(s_0 \times t)=x$ for all $0 \le t \le 1$. One can also define $\pi_k(X,x)$ as the set of homotopy classes of maps from the standard k-cube I^k to X that map the boundary of the cube to x, with homotopies also mapping the boundary to x throughout.

Exercise 22.14. (a) Show that these two definitions agree by showing that S^k is homeomorphic to the space obtained by identifying all points of the boundary of I^k to a point. (b) Show that $\pi_k(X,x)=0$ for all $k>0$ if X is contractible. (c) Show that a map $f\colon X \to Y$ determines maps $f_*\colon \pi_k(X,x) \to \pi_k(Y,f(x))$, which are functorial, and that maps that are homotopic through basepoint-preserving homotopies determine the same map on homotopy groups. (d) Show that $\pi_k(S^n, s_0)=0$ for $0<k<n$.

The sets $\pi_k(X,x)$ can be made into groups, much as for the fundamental group. Using the definition by cubes, one can "multiply" two maps Γ and Λ from I^k to X, defining $\Gamma \cdot \Lambda$ by using the first coordinate:

$$\Gamma \cdot \Lambda(t_1, \ldots, t_k) = \begin{cases} \Gamma(2t_1, t_2, \ldots, t_k), & 0 \le t_1 \le 1/2, \\ \Lambda(2t_1-1, t_2, \ldots, t_k), & 1/2 \le t_1 \le 1. \end{cases}$$

Exercise 22.15. (a) Show that this operation is well defined on homotopy classes, and makes $\pi_k(X,x)$ into a group. (b) Show that the maps f_* of Exercise 22.14 are homomorphisms of groups.

Problem 22.16. Show that, for all $k>1$, the group $\pi_k(X,x)$ is abelian.

It is a fact that $\pi_n(S^n, s_0) \cong \mathbb{Z}$, although this is quite a bit harder to prove. Note that this gives a strong notion of degree for maps of S^n to S^n: it defines the degree, and shows that maps are classified up to homotopy by their degree. In the next chapter we will use chains to define homology groups $H_k(X)$, which are easier to calculate, and we will show that $H_n(S^n) = \mathbb{Z}$ and $H_k(S^n) = 0$ for all $k > 0$, $k \neq n$. In stark contrast with the homology groups, for $k > n$, the groups $\pi_k(S^n, s_0)$ need not be trivial.

Exercise 22.17. Identifying S^3 with $\{(z, w) \in \mathbb{C}^2: |z|^2 + |w|^2 = 1\}$, show that the map that takes (z, w) to $w/z \in \mathbb{C} \subset S^2$ determines a continuous mapping from S^3 to S^2.

It is a fact that $\pi_3(S^2, s_0) \cong \mathbb{Z}$, with generator given by the mapping of the preceding exercise, which is called a *Hopf* mapping (see Hilton (1961)). We will show in the next chapter that the fibers of the Hopf mapping are circles that are "linked" together in S^3, which is at least an indication of the nontriviality of the Hopf map. It should be pointed out, however, that in spite of enormous effort, which have produced calculations of many special cases, the groups $\pi_k(S^n, s_0)$ are far from known in general.

22d. Higher De Rham Cohomology

All of the discussion of §22a generalizes to n variables, a k-form being an expression

$$\Sigma f_{i_1 i_2 \ldots i_k} dx_{i_1} dx_{i_2} \ldots dx_{i_k} = \Sigma f_I \, dx_I$$

the sum over all $1 \leq i_1 < \ldots < i_k \leq n$, with the coefficients $f_{i_1 i_2 \ldots i_k}$ $\mathscr{C}^\infty$ functions on an open set in $\mathbb{R}^n$. There is a differential d that takes a k-form to a $(k+1)$-form, and again $d \circ d = 0$. For U open in $\mathbb{R}^n$ one then gets De Rham groups $H^k U$, the space of closed k-forms modulo the space of exact k-forms. The main complication is in keeping track of the signs. This is best done by introducing formally the "exterior algebra" structure that is already apparent in the plane and 3-space: one allows the differentials to be written in arbitrary order, but put in a sign whenever they are interchanged: $dy\,dx = -dx\,dy$, together setting $dx\,dx = 0$. (The usual notation for this exterior product is the wedge "$\wedge$," so one writes $dx \wedge dy$ in place of our $dx\,dy$.) Working this out properly belongs in an advanced calculus course.

As with surfaces, one can define k-forms on an arbitrary differentiable manifold as collections of k-forms on the coordinate neighborhoods of a chart that transform properly on overlaps. If you know about differential forms on a manifold, it is not difficult to generalize the idea of De Rham cohomology. Again there are differentials d from k-forms to $(k+1)$-forms, with $d \circ d = 0$. In this section we will assume familiarity with notions of manifolds and differential forms. If this applies to you, fine; if not, you can either stick to low dimensions where we have done it by hand, or assume this formalism in general—or you can turn immediately to the next chapter, which does not depend on any of this.

If X is a differentiable manifold, one can define the De Rham cohomology group H^kX as the vector space of closed k-forms modulo the subspace of exact k-forms. If $f\colon X \to Y$ is a differentiable map, there is a notion of pull-back of k-forms from Y to X, taking ω on Y to $f^*\omega$ on X. This commutes with the differential, so determines a (functorial) homomorphism $f^*\colon H^kY \to H^kX$.

Exercise 22.18. If X is a disjoint union of a finite or infinite number of manifolds X_i, show that H^kX is the direct product of the H^kX_i, i.e., specifying a class on X is the same as specifying a class on each X_i.

The Mayer–Vietoris exact sequence is defined with almost no change from the case with H^0 and H^1. To define

$$\delta\colon H^k(U \cap V) \to H^{k+1}(U \cup V),$$

as before, one uses a partition of unity subordinate to an open covering $X = U \cup V$ by U and V to write a closed k-form ω on $U \cap V$ as the difference $\mu_1 - \mu_2$ of a k-form μ_1 on U and a k-form μ_2 on V; there is a closed $(k+1)$-form on $U \cup V$ that is $d\mu_1$ on U and $d\mu_2$ on V, and this $(k+1)$-form represents the image in $H^{k+1}(U \cup V)$ of the class represented by ω in $H^k(U \cap V)$. As before (and see §24a), one proves:

Mayer–Vietoris Theorem 22.19. *For any open sets U and V in a manifold of dimension n, there is an exact sequence*

$$\begin{aligned}
0 \longrightarrow H^0(U \cup V) &\xrightarrow{+} H^0U \oplus H^0V \xrightarrow{-} H^0(U \cap V)\\
\xrightarrow{\delta} H^1(U \cup V) &\xrightarrow{+} H^1U \oplus H^1V \xrightarrow{-} H^1(U \cap V) \xrightarrow{\delta}\\
&\cdots \longrightarrow \cdots\\
\xrightarrow{\delta} H^n(U \cup V) &\xrightarrow{+} H^nU \oplus H^nV \xrightarrow{-} H^n(U \cap V) \longrightarrow 0
\end{aligned}$$

To calculate these groups, one needs in addition the

Poincaré Lemma 22.20. *If $p\colon X \times \mathbb{R} \to X$ is the projection, then $p^*\colon H^kX \to H^k(X \times \mathbb{R})$ is an isomorphism.*

The inverse isomorphism to p^* is $s^*\colon H^k(X\times\mathbb{R})\to H^k(X)$, where s is the embedding $x\mapsto x\times 0$ of X in $X\times\mathbb{R}$. The problem is to show that $p^*\circ s^* = (s\circ p)^*$ is the identity. The idea of the proof is to construct a linear map H from the space of k-forms on $X\times\mathbb{R}$ to the space of $(k-1)$-forms on $X\times\mathbb{R}$, for each k, such that for any form ω on $X\times\mathbb{R}$,

$$\omega - p^*\circ s^*(\omega) = d(H(\omega)) + H(d(\omega)). \tag{22.21}$$

(Note that the two H's and the two d's in this equation are defined on different spaces of forms!) It follows that if ω is closed, then $\omega - p^*\circ s^*(\omega) = d(H(\omega))$, so ω and $p^*\circ s^*(\omega)$ define the same De Rham cohomology class.

Problem 22.22. Show that any k-form ω on $X\times\mathbb{R}$ has a unique expression as a sum of a k-form not involving dt, where t is the coordinate on $\mathbb{R}$, and one of the form $dt\wedge\mu$, where μ is, in local coordinates, a sum of expressions $f\cdot dx_I$, with the x_i coordinates on X, and f is a function on the product of the coordinate neighborhood with $\mathbb{R}$. Define H of such a form to be the form obtained by integrating μ with respect to the variable in $\mathbb{R}$ (so forms not involving dt are mapped to 0). For example, if $X\subset\mathbb{R}^n$, and μ is the form $f\,dx_I$, then $H(dt\wedge\mu)$ is the form $g\,dx_I$, where

$$g(x_1,\ldots,x_n,t) = \int_0^t f(x_1,\ldots,x_n,s)\,ds.$$

Show that this operator is well defined and satisfies (22.21).

For any real number t, if $s_t\colon X\to X\times\mathbb{R}$ maps x to $x\times t$, then, since $p\circ s_t$ is the identity, it follows that $s_t^*\colon H^k(X\times\mathbb{R})\to H^k(X)$ is the inverse to $p^*\colon H^k(X)\to H^k(X\times\mathbb{R})$. In particular, the maps s_t^* are the same for all t. This implies that if $F\colon X\times\mathbb{R}\to Y$ is differentiable, all the maps $F_t\colon X\to Y$, $F_t(x) = F(x\times t)$, determine the same maps $F_t^*\colon H^kY\to H^kX$. Indeed, $F_t = F\circ s_t$, so $F_t^* = s_t^*\circ F^*$ is independent of t.

Problem 22.23. (a) Use the Poincaré lemma and Mayer–Vietoris to calculate the De Rham cohomology of $\mathbb{R}^n$, S^n, and $\mathbb{R}^n\setminus\{0\}$. (b) Show that the $(n-1)$-form ω_{n-1} defined on $\mathbb{R}^n\setminus\{0\}$ by

$$\omega_{n-1} = \frac{\sum_{i=1}^{n}(-1)^{i-1}x_i\,dx_1\wedge\ldots\wedge dx_{i-1}\wedge dx_{i+1}\wedge\ldots\wedge dx_n}{(x_1^2+\ldots+x_n^2)^{n/2}}$$

is closed and gives a generator of $H^{n-1}(\mathbb{R}^n \setminus \{0\})$. If $f\colon S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ is differentiable, this can be used to define a higher-dimensional winding number, or "engulfing number": $W(f, 0)$ is the integral of $f^*\omega_{n-1}$ over S^{n-1}, divided by the integral of ω_{n-1} over S^{n-1}.

22e. Cohomology with Compact Supports

There is another way to use differential forms to construct cohomology groups, for open sets in $\mathbb{R}^n$ or any $\mathscr{C}^\infty$ manifolds, which we sketch briefly here. These cohomology groups are called *De Rham groups with compact supports*, and denoted $H^k_c X$, the subscript c standing for "compact." These are defined exactly as for the ordinary De Rham groups, but using differential forms with compact support, i.e., forms for which there is some compact set K contained in X such that the form vanishes identically outside K. Define $H^k_c X$ to be the quotient space of the closed k-forms with compact support by the subspace of forms that are differentials of $(k-1)$-forms with compact support.

If X is compact, of course, all forms have compact support, so $H^k_c X = H^k X$. In spite of this and the similarity of definition, however, these groups are quite different on noncompact manifolds. For example, locally constant functions on a noncompact connected space can never have compact support unless they are identically zero:

$$H^0_c X = 0 \quad \text{if } X \text{ is connected and not compact}.$$

In fact, we will see that the theories H^k and H^k_c behave in an opposite, or dual, way. By using a partition of unity as in Chapter 18, if X is an oriented n-manifold, one can integrate an n-form with compact support over the whole manifold. Using Stokes' theorem, it follows similarly that the integral of the differential of a closed $(n-1)$-form with compact support is zero, so one has a map

$$H^n_c X \to \mathbb{R}, \qquad \omega \mapsto \int_X \omega.$$

It is easy to produce n-forms that are positive on a small piece of a coordinate neighborhood, and zero elsewhere, to see that this map is not zero. We will see in Chapter 24 that, if X is connected as well, then this map is an isomorphism.

Exercise 22.24. If X is a disjoint union of a finite or infinite number of open sets X_i, show that $H^k_c X$ is the direct sum of the $H^k_c X_i$, i.e., specifying a class on X is the same as specifying a class on each X_i, except that all but a finite number must be zero.

There is a Mayer–Vietoris exact sequence for cohomology with compact supports, but it is different from that without supports. First of all, there are no restriction maps, since if ω has compact support on an open set, its restriction to an open subset may no longer have compact support. In fact, the maps go the other way: if U_1 is an open subset of U_2, any k-form ω with compact support on U_1 extends by zero outside U_1 to define a k-form with (the same) compact support on U_2. Since any point in $U_2 \setminus U_1$ has a neighborhood not meeting the support of ω, this extension is $\mathscr{C}^\infty$. This extension commutes with the differential map d, so determines a linear map

$$H^k_c(U_1) \to H^k_c(U_2).$$

In particular, for two open sets U and V, we have diagrams

$$\begin{tikzcd} & H^k_c(U) \arrow[dr] & \\ H^k_c(U \cap V) \arrow[ur] \arrow[dr] & & H^k_c(U \cup V). \\ & H^k_c(V) \arrow[ur] & \end{tikzcd}$$

We want to define a coboundary map $\delta: H^k_c(U \cup V) \to H^{k+1}_c(U \cap V)$. Given a class in $H^k_c(U \cap V)$, represent it by a closed k-form ω with compact supports. We can write ω as a sum $\omega_1 + \omega_2$, with ω_1 and ω_2 k-forms with compact supports in U and V, respectively. In fact, if $\psi_1 + \psi_2 = 1$ is a partition of unity subordinate to U and V, then $\omega_1 = \psi_1 \cdot \omega$ and $\omega_2 = \psi_2 \cdot \omega$ are such forms. Let η be the $(k+1)$-form on $U \cap V$ that is (the restriction of) $d\omega_1$. From the equation $0 = d\omega = d\omega_1 + d\omega_2$, we have $d\omega_1 = -d\omega_2$ on $U \cap V$, so the support of η is contained in the intersection of the supports of ω_1 and ω_2. In particular, η has compact support on $U \cap V$. Clearly η is closed. Set

$$\delta([\omega]) = [\eta].$$

It is not hard to verify that this is independent of the choice, and it is the familiar (by now) argument (and see §24a) to prove the

Mayer–Vietoris Theorem 22.25 (Compact Supports). *For open sets U and V in an n-dimensional manifold, there is an exact sequence*

$$
\begin{array}{l}
0 \longrightarrow H_c^0(U\cap V) \xrightarrow{+} H_c^0U \oplus H_c^0V \xrightarrow{-} H_c^0(U\cup V) \\
\xrightarrow{\delta} H_c^1(U\cap V) \xrightarrow{+} H_c^1U \oplus H_c^1V \xrightarrow{-} H_c^1(U\cup V) \xrightarrow{\delta} \\
\qquad\qquad\qquad \cdots \longrightarrow \cdots \\
\xrightarrow{\delta} H_c^n(U\cap V) \xrightarrow{+} H_c^nU \oplus H_c^nV \xrightarrow{-} H_c^n(U\cup V) \longrightarrow 0.
\end{array}
$$

As before, to complete the basic tools for calculating these groups, we need to compare a manifold X and $X \times \mathbb{R}$. This time the projection p from $X \times \mathbb{R}$ to X determines homomorphisms

$$p_*\colon H_c^k(X\times\mathbb{R}) \to H_c^{k-1}(X),$$

by "integrating along the fiber," as follows. As in Problem 22.22, one can write a k-form with compact support of $X \times \mathbb{R}$ as a sum of a form not involving dt, and a k-form $dt\wedge\mu$, where μ is, in local coordinates, a sum of expressions $f\cdot dx_I$, with the x_i coordinates on X, and f a function on $X \times \mathbb{R}$. Define p_* of such a form $dt\wedge f\cdot dx_I$ to be $g\cdot dx_I$, where

$$g(x_1, \ldots, x_n) = \int_{-\infty}^{\infty} f(x_1, \ldots, x_n, t)\,dt$$

(so forms not involving dt are mapped to zero). Note that these integrals are really over finite intervals, by the assumption of compact support. One checks that this is well defined, and that $p_*(d\omega) = d(p_*\omega)$, so p_* determines a map on cohomology classes as indicated. If $s\colon X\to X\times\mathbb{R}$ is the inclusion $x\mapsto x\times 0$, there is a map

$$s_*\colon H_c^{k-1}(X) \to H_c^k(X\times\mathbb{R})$$

determined by sending a form ω to $\rho(t)\,dt\wedge\omega$, where ρ is any function with compact support on $\mathbb{R}$ such that $\int_{-\infty}^{\infty}\rho(t)\,dt = 1$. One checks that this commutes with d, so defines a map on cohomology, and that $p_*\circ s_*$ is the identity.

Poincaré Lemma 22.26 (Compact Supports). *For any manifold X, $p_*\colon H_c^k(X\times\mathbb{R})\to H_c^{k-1}(X)$ is an isomorphism.*

Problem 22.27. Prove this by constructing an operator H from k-forms with compact support to $(k-1)$-forms with compact support on $X\times\mathbb{R}$. This operator should vanish on forms without "dt," and take $dt\wedge\mu$, where μ is, in local coordinates, a sum of expressions

$f \cdot dx_I$, to $g \cdot \mu$, with

$$g(x_1, \ldots, x_n, t) = \int_{-\infty}^{t} f(x_1, \ldots, x_n, s)\, ds - \int_{-\infty}^{t} \rho(s)\, ds \int_{-\infty}^{\infty} f(x_1, \ldots, x_n, s)\, ds .$$

Show that for any k-form ω with compact support, $\omega - s_* p_* \omega = d(H(\omega)) + H(d(\omega))$, and deduce that p_* and s_* determine inverse isomorphisms.

Exercise 22.28. Calculate $H_c^k X$, when X is $\mathbb{R}^n$, S^n, and $\mathbb{R}^n \setminus \{0\}$.

CHAPTER 23

Higher Homology

23a. Homology Groups

The groups H_0X and H_1X are the beginning of a series of abelian groups H_kX, defined for any topological space X. Define a *k-cube* in X to be a continuous map $\Gamma: I^k \to X$, where I^k is the k-dimensional cube, i.e., $I = [0, 1]$, so

$$I^k = [0,1] \times \ldots \times [0,1] \subset \mathbb{R}^k.$$

For any such mapping Γ, and any integer i between 1 and k, and any $0 \leq s \leq 1$, define a $(k-1)$-cube $\partial_i^s\Gamma$, which is obtained by restricting Γ to the slice of the ith coordinate at s:

$$\partial_i^s\Gamma: I^{k-1} \to X, \quad \partial_i^s\Gamma(t_1, \ldots, t_{k-1}) = \Gamma(t_1, \ldots, t_{i-1}, s, t_i, \ldots, t_{k-1}).$$

Call Γ *degenerate* if, for some i, $\partial_i^s\Gamma$ is a constant function of s, and *nondegenerate* otherwise. (When $k = 1$, Γ is a path, and degenerate is the same as a constant path.) By convention, $I^0 = \{0\}$, so as 0-cube is given by a point in X; no 0-cube is regarded as degenerate.

Let C_kX be the free abelian group on the nondegenerate k-cubes in X, so an element of C_kX is a finite linear combination $\Sigma n_\alpha\Gamma_\alpha$, with Γ_α a k-cube in X and n_α an integer. An element of C_kX is called a cubical *k-chain* on X. It is useful to regard any finite linear combination $\Sigma n_\alpha\Gamma_\alpha$ of arbitrary k-cubes as an element of C_kX, by simply discarding any degenerate k-cube Γ_α that appears. (In other words, C_kX is identified with the quotient of the free abelian group on all k-cubes in X, modulo the subgroup generated by degenerate k-cubes.)

If $\Gamma\colon I^k \to X$ is a k-cube in X, its *boundary* $\partial\Gamma$ in $C_{k-1}X$ is defined by the formula

$$\partial\Gamma = \sum_{i=1}^{k} (-1)^i(\partial_i^0\Gamma - \partial_i^1\Gamma),$$

which is the sum of the $2k$ faces of Γ, each with a coefficient of $+1$ or -1. (Note that, even if Γ is nondegenerate, some of the $\partial_i^0\Gamma$ or $\partial_i^1\Gamma$ occurring in the formula can be degenerate, so they are discarded.) This is extended linearly to a homomorphism

$$\partial\colon C_kX \to C_{k-1}X$$

by the formula $\partial(\Sigma n_\alpha\Gamma_\alpha) = \Sigma n_\alpha(\partial\Gamma_\alpha)$. A k-chain is called a *k-cycle* if its boundary is zero; the k-cycles form a subgroup Z_kX of C_kX. The boundaries of $(k+1)$-chains form a subgroup B_kX of C_kX.

Exercise 23.1. Show that for any $(k+1)$-cube Γ, $\partial(\partial\Gamma) = 0$ in $C_{k-1}X$.

From this exercise it follows that B_kX is a subgroup of Z_kX, so we can define the kth homology group of X to be quotient

$$H_kX = Z_kX/B_kX.$$

Exercise 23.2. (a) Show that $H_kX = 0$ if X is a point and $k > 0$. (Note that this would not be true if degenerate cubes had not been discarded.) (b) Verify that for $k = 0$ and $k = 1$, these are the groups we studied in Chapter 6. (c) Show how any continuous mapping $f\colon X \to Y$ determines homomorphisms $f_*\colon H_kX \to H_kY$, and show that these are functorial in the sense of Exercise 6.20. (d) Construct homomorphisms from $\pi_k(X, x)$ to H_kX that are compatible with the maps of (c) and Exercise 22.14.

Proposition 23.3. *If f and g are homotopic maps from X to Y, then f_* and g_* determine the same homomorphisms from H_kX to H_kY.*

Proof. Suppose $H\colon X \times [0, 1] \to Y$ is a homotopy from f to g, and $\Gamma\colon I^k \to X$ is a k-cube; define a $(k+1)$-cube $R(\Gamma)$ by the formula

$$R(\Gamma)(s, t_1, \ldots, t_k) = H(\Gamma(t_1, \ldots, t_k) \times s).$$

If $\Sigma n_\alpha\Gamma_\alpha$ is a k-cycle, a little calculation shows that the boundary of $\Sigma n_\alpha R(\Gamma_\alpha)$ is $\Sigma n_\alpha(f \circ \Gamma_\alpha) - \Sigma n_\alpha(g \circ \Gamma_\alpha)$, which completes the proof. A more elegant way to see this is to extend R by linearity to a map $R\colon C_kX \to C_{k+1}X$, $R(\Sigma n_\alpha\Gamma_\alpha) = \Sigma n_\alpha R(\Gamma_\alpha)$. Then a formal calculation shows that

$$g_* - f_* = \partial \circ R + R \circ \partial$$

as homomorphisms from C_kX to C_kX. Then if z is a k-cycle,

$$g_*(z) - f_*(z) = \partial \circ R(z) + R \circ \partial(z) = \partial(R(z)),$$

which shows that $g_*(z)$ and $f_*(z)$ differ by a boundary. □

It follows from the proposition that if $Y \subset X$ is a deformation retract, then $H_kY \to H_kX$ is an isomorphism for all k. For example, if X is contractible, then $H_kX = 0$ for all $k > 0$.

23b. Mayer–Vietoris for Homology

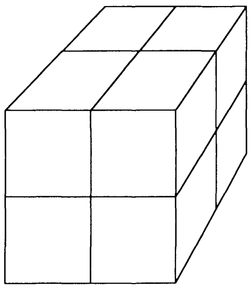
To calculate the higher homology groups of more interesting spaces, we want to extend the Mayer–Vietoris sequence to these higher groups: if U and V are open subsets of a space, there is an exact sequence

$$\cdots \xrightarrow{\partial} H_k(U \cap V) \xrightarrow{-} H_kU \oplus H_0V \xrightarrow{+} H_k(U \cup V)$$

$$\xrightarrow{\partial} H_{k-1}(U \cap V) \xrightarrow{-} H_{k-1}U \oplus H_{k-1}V \xrightarrow{+} H_{k-1}(U \cup V) \xrightarrow{\partial} \cdots .$$

The idea is very much as in Chapter 10. The key is the definition of the "boundary maps" $\partial\colon H_k(U \cup V) \to H_{k-1}(U \cap V)$. For this, we show that any class in $H_k(U \cup V)$ can be represented by a k-cycle z that is a sum of a chain c_1 on U and a chain c_2 on V. Then $\partial c_1 = -\partial c_2$ is a cycle on $U \cap V$, and this represents the image class in $H_{k-1}(U \cap V)$.

To carry out the construction of the boundary map, we need a systematic way to subdivide k-cubes and chains into sums of smaller chains, generalizing the constructions we used in Chapter 10. Given a k-cube $\Gamma\colon I^k \to X$, we define $S(\Gamma)$ to be the sum of the 2^k k-cubes obtained by restricting Γ to each of the 2^k subcubes obtained by subdividing the cube:



Each of these restrictions must be renormalized, to be defined on the cube I^k with sides of length 1. In symbols,

$$S(\Gamma) = \Sigma\Gamma_{\varepsilon_1,\ldots,\varepsilon_k},$$

the sum over all 2^k choices of $\varepsilon_i = 0$ or 1, and $\Gamma_{\varepsilon_1,\ldots,\varepsilon_k}$ is the k-cube defined by the formula

$$\Gamma_{\varepsilon_1,\ldots,\varepsilon_k}(t_1, \ldots, t_k) = \Gamma(1/2(t_1+\varepsilon_1), \ldots, 1/2(t_k+\varepsilon_k)).$$

This is extended by linearity to give a homomorphism $S\colon C_kX \to C_kX$, i.e., by defining $S(\Sigma n_\alpha \Gamma_\alpha) = \Sigma n_\alpha S(\Gamma_\alpha)$. If the boundary of $S(\Gamma)$ is calculated, all the terms corresponding to inner faces cancel, and one gets the result of subdividing the boundary of Γ. In symbols,

$$\partial \circ S = S \circ \partial, \tag{23.4}$$

as homomorphisms from C_kX to $C_{k-1}X$.

Exercise 23.5. Verify this formula.

We can iterate this subdivision operation, defining for any k-chain c new k-chains $S(c)$, $S^2(c) = S(S(c))$, $S^3(c) = S(S(S(c)))$, and so on.

Lemma 23.6. *If X is a union of two open sets U and V, and c is a k-chain on X, then, for sufficiently large p, $S^p(c)$ can be written as a sum $c_1 + c_2$ where c_1 is a k-chain on U and c_2 is a k-chain on V.*

Proof. This is an immediate consequence of the Lebesgue lemma, since, if $\Gamma\colon I^k \to X$ is a k-cube, the image of each k-cube appearing in $S^p(\Gamma)$ is the image of a subcube of I^k with sides of length $1/2^p$. □

To use this construction and lemma, we want to know that, if z is a k-cycle, then $S(z)$ is a k-cycle defining the same homology class as z. For this we proceed as follows. Let $\alpha\colon [0,1] \to [0,1]$ be defined by the formula

$$\alpha(t) = \begin{cases} 2t, & 0 \le t \le 1/2, \\ 1, & 1/2 \le t \le 1. \end{cases}$$

If Γ is a k-cube in X, define a k-cube $A(\Gamma)$ by the formula

$$A(\Gamma)(t_1, \ldots, t_k) = \Gamma(\alpha(t_1), \ldots, \alpha(t_k)).$$

Note that the "first corner" $A(\Gamma)_{0,\ldots,0}$ of this k-cube is Γ, and all the other $A(\Gamma)_{\varepsilon_1,\ldots,\varepsilon_k}$ are degenerate. Extend this by linearity as usual to a homomorphism $A\colon C_kX \to C_kX$. (If $k=0$, define A to be the identity map.) By the observation just made, we have

$$S \circ A = I, \tag{23.7}$$

where $I\colon C_kX \to C_kX$ is the identity map. Now define, for a k-cube Γ, a $(k+1)$-cube $H(\Gamma)$ by the formula

$$H(\Gamma)(s, t_1, \ldots, t_k) = \Gamma((1-s)\alpha(t_1) + st_1, \ldots, (1-s)\alpha(t_k) + st_k),$$

and extend by linearity to a homomorphism $H\colon C_kX \to C_{k+1}X$. (If $k=0$, set $H=0$.) Note that $\partial_1^0(H(\Gamma)) = A(\Gamma)$ and $\partial_1^1(H(\Gamma)) = \Gamma$. From this one sees that

$$\partial \circ H + H \circ \partial = I - A, \tag{23.8}$$

as homomorphisms from C_kX to C_kX.

Exercise 23.9. Verify this formula.

For each $p \geq 1$, define a homomorphism $R_p\colon C_kX \to C_{k+1}X$ by the formula

$$R_p = S \circ H \circ (I + S + S^2 + \ldots + S^{p-1}).$$

Then we have, for all $p \geq 1$, the identity

$$\partial \circ R_p + R_p \circ \partial = S^p - I. \tag{23.10}$$

In fact, this is a formal calculation, following from (23.4), (23.7), and (23.8), as follows. When $p=1$, $R_1 = S \circ H$, and

$$\begin{aligned}
\partial \circ R_1 + R_1 \circ \partial &= \partial \circ S \circ H + S \circ H \circ \partial \\
&= S \circ \partial \circ H + S \circ H \circ \partial \\
&= S \circ (\partial \circ H + H \circ \partial) = S \circ (I - A) \\
&= S - S \circ A = S - I.
\end{aligned}$$

For $p > 1$, $R_p = R_1 \circ S_p$, where $S_p = I + S + \ldots + S^{p-1}$. We use the case $p=1$ in the form $\partial \circ R_1 = S - I - R_1 \circ \partial$, and we use the fact that ∂ commutes with S_p by (23.4), together with the identity $S^p - I = (S - I) \circ S_p$. Calculating, we have

$$\begin{aligned}
\partial \circ R_p + R_p \circ \partial &= \partial \circ R_1 \circ S_p + R_p \circ \partial \\
&= (S - I - R_1 \circ \partial) \circ S_p + R_p \circ \partial \\
&= (S - I) \circ S_p + (-R_1 \circ \partial) \circ S_p + R_p \circ \partial \\
&= S^p - I - R_1 \circ S_p \circ \partial + R_p \circ \partial \\
&= S^p - I - R_p \circ \partial + R_p \circ \partial = S^p - I,
\end{aligned}$$

as asserted.

Now suppose that $X = U \cup V$. The definition of the boundary homomorphism from $H_k(U \cup V)$ to $H_{k-1}(U \cap V)$ depends on the following lemma:

Lemma 23.11. *(a) Any homology class in H_kX can be represented by a cycle z on X of the form $z = c_1 + c_2$, where c_1 is a k-chain on U and c_2 is a k-chain on V. (b) The $(k-1)$-chain $\partial c_1 = -\partial c_2$ is a cycle on $U \cap V$, and its homology class in $H_{k-1}(U \cap V)$ is independent of choice of c_1 and c_2.*

Proof. For (a), take any cycle c that represents the homology class. By Lemma 23.6, for some $p \geq 1$, the chain $S^p c$ can be written as the sum of a chain c_1 on U and a chain c_2 on V. By (23.10),

$$S^p c - c = \partial(R_p(c)) + R_p(\partial(c)) = \partial(R_p(c)),$$

from which it follows that $z = S^p c$ is a cycle representing the same homology class as c.

For (b), suppose $z' = c_1' + c_2'$ is another representative of the same form for the same homology class. There is a $(k+1)$-chain w on X with $\partial(w) = z' - z$. By Lemma 23.6, there is a $p \geq 1$ such that $S^p(w)$ can be written as a sum of a chain on U and a chain on V. Applying (23.10) to the chain $\partial(w)$, we have

$$\begin{aligned} z' - z = \partial(w) &= S^p(\partial(w)) - R_p(\partial(\partial(w))) - \partial(R_p(z' - z)) \\ &= \partial(S^p(w)) - \partial(R_p(z' - z)) = \partial(S^p(w) - R_p(z' - z)). \end{aligned}$$

From the formula for R_p it follows that R_p takes a chain on U to a chain on U and a chain on V to a chain on V. We know that $z' - z$ is a sum of a chain on U and a chain on V, and it follows that $R_p(z' - z)$ is also. It follows that there are $(k+1)$-chains y_1 and y_2 on U and V such that $S^p(w) - R_p(z' - z) = y_1 + y_2$, so

$$z' - z = \partial(y_1 + y_2).$$

This means that we have an equality of k-chains

$$c_1' - c_1 - \partial(y_1) = -(c_2' - c_2 - \partial(y_2)),$$

the left side of which is a chain on U and the right side is a chain on V. This chain, denoted x, is a chain on $U \cap V$, and it follows that

$$\partial(x) = \partial(c_1') - \partial(c_1),$$

so the cycles $\partial(c_1')$ and $\partial(c_1)$ differ by a boundary on $U \cap V$, as asserted. □

Define $\partial \colon H_kX \to H_{k-1}(U \cap V)$ by taking the homology class $[z]$ of a cycle z of the form $c_1 + c_2$ with c_1 and c_2 chains on U and V, to the homology class $[\partial(c_1)]$ of the cycle $\partial(c_1) = -\partial(c_2)$ on $U \cap V$. It follows from Lemma 23.11 that this definition makes sense. The proof that it is a homomorphism, and that the resulting Mayer–Vietoris sequence is exact, is precisely the same as in Chapter 10, so will not be repeated; the general algebra for this will be described in §24a.

In fact, the above argument shows something more. Let X be any space, and $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ any open covering of X. Let $C_k(X)^{\mathcal{U}}$ be the subgroup of $C_k(X)$ consisting of linear combinations of nondegenerate k-cubes $\Gamma: I^k \to X$ such that the image of Γ is contained in one (or more) of the open sets U_α. These cubes are said to be *small* with respect to $\mathcal{U}$. The boundary operator maps $C_k(X)^{\mathcal{U}}$ to $C_{k-1}(X)^{\mathcal{U}}$, so we can define the corresponding homology groups $H_k(X)^{\mathcal{U}}$. There is a natural map from $H_k(X)^{\mathcal{U}}$ to $H_k(X)$. The following proposition says that we can always calculate homology by using chains that are small with respect to any convenient covering.

Proposition 23.12. *The natural map $H_k(X)^{\mathcal{U}} \to H_k(X)$ is an isomorphism.*

Exercise 23.13. Use (23.4), (23.7), (23.8), and (23.10) to prove this proposition.

This proposition also gives a more concise way to construct the Mayer–Vietoris exact sequence:

Exercise 23.14. For $\mathcal{U} = \{U, V\}$, construct a homomorphism from $H_k(X)^{\mathcal{U}}$ to $H_{k-1}(U \cap V)$, and show that there is a long exact sequence

$$\ldots \xrightarrow{\partial} H_k(U \cap V) \xrightarrow{-} H_k U \oplus H_k V \xrightarrow{+} H_k(X)^{\mathcal{U}} \xrightarrow{\partial} H_{k-1}(U \cap V) \to \ldots .$$

Combine with the proposition to get the full Mayer–Vietoris sequence.

The following is a useful general consequence of the Mayer–Vietoris sequence.

Exercise 23.15. Suppose a space X is a union of some open sets $U_1, \ldots, U_p$ such that all homology groups $H_k(Y)$ vanish for any intersection $Y = U_{i_1} \cap \ldots \cap U_{i_r}$ of these open sets and all $k > 0$. (a) Show that $H_k(X) = 0$ for $k \geq p$. (b) If, in addition, each intersection Y is connected, and $p \geq 2$, show that $H_{p-1}(X) = 0$. (c) Finally, if each intersection Y is connected and nonempty, show that $H_k(X) = 0$ for all $k > 0$.

23c. Spheres and Degree

We saw that S^n is simply connected if $n \geq 2$, so $H_1 S^n = 0$. With Mayer–Vietoris, one can calculate the homology groups of all spheres S^n. One can cover S^n by two open sets U and V each homeomorphic to open disks in $\mathbb{R}^n$, whose intersection is homeomorphic to $S^{n-1} \times I^\circ$ for an open interval I°. Mayer–Vietoris gives an isomorphism

$$\partial \colon H_k(S^n) \xrightarrow{\cong} H_{k-1}(S^{n-1} \times I^\circ) \cong H_{k-1}(S^{n-1})$$

for all $k > 1$. From this one sees that

$$H_k(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } n, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 23.16. Show that the complements of the south and north poles $(0, \ldots, 0, -1)$ and $(0, \ldots, 0, 1)$ satisfy the conditions for U and V, and use Mayer–Vietoris to complete the proof of this calculation.

Equipped with homology groups, the definition of the degree of a continuous map $f \colon S^n \to S^n$ is easy: Since $H_n S^n \cong \mathbb{Z}$, the induced map $f_* \colon H_n S^n \to H_n S^n$ is multiplication by an integer, and this integer is defined to be the degree of f, denoted $\deg(f)$. Equivalently, if $[z]$ is a generator for $H_n S^n$, $\deg(f)$ is the integer such that $f_*([z]) = \deg(f) \cdot [z]$. (Note that using the other generator $-[z]$ leads to the same degree.)

Exercise 23.17. (a) Show that homotopic maps from S^n to S^n have the same degree. (b) Show that if $f \colon S^n \to S^n$ extends to a continuous map from D^{n+1} to S^n, then $\deg(f) = 0$. (c) Show that if f and g are maps from S^n to S^n, then $\deg(g \circ f) = \deg(g) \cdot \deg(f)$. (d) Show that for any integer d and any $n \geq 1$ there are maps $f \colon S^n \to S^n$ of degree d.

In fact the converses of (a) and (b) of the preceding exercise are true, but more difficult.

Having the notion of degree, we can define the generalization of winding number: the *engulfing number* $W(f, P)$ of a continuous map $f \colon S^{n-1} \to \mathbb{R}^n \setminus \{P\}$ *around* P. This can be defined to be the degree of the map that follows f by projection onto a sphere around P, i.e., define $W(f, P)$ to be the degree of the map

$$S^{n-1} \to S^{n-1}, \qquad x \mapsto \frac{f(x) - P}{\|f(x) - P\|}.$$

Problem 23.18. (a) Show that, as a function of P, this number is constant on connected components of $\mathbb{R}^n \setminus f(S^{n-1})$. (b) State and prove an n-dimensional analogue of the dog-on-a-leash theorem. (c) Show that for $f\colon S^n \to \mathbb{R}^n \setminus \{0\}$ differentiable, this definition agrees with that in Problem 22.23.

Similarly, one can define the *local degree* of a continuous mapping $f\colon U \to V$ between open sets in $\mathbb{R}^n$ at a point P in U, provided there is a neighborhood U_P of P such that no other point of U_P has the same image as P. This is the degree of the mapping

$$S^{n-1} \to S^{n-1}, \qquad x \mapsto \frac{f(P+rx)-f(P)}{\|f(P+rx)-f(P)\|},$$

for any positive r so that U_P contains the ball of radius r around P. This can be used for example to define the notion of a homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ (or S^n to S^n, or X to X for any oriented manifold) being *orientation preserving* or *orientation reversing*, according as the local degree at any point is $+1$ or -1.

Exercise 23.19. (a) Show that this local degree is a continuous function of the point, so the notion of orientation preserving or reversing is well defined. (b) Show that a homeomorphism of S^n is orientation preserving or reversing according as its degree is $+1$ or -1. Show that a homeomorphism of $\mathbb{R}^n$ extends to a homeomorphism of $S^n = \mathbb{R}^n \cup \{\infty\}$, whose degree therefore determines whether the original map is orientation preserving. (c) For a diffeomorphism, show that the local degree is given by the sign of the determinant of the Jacobian.

With the concept of degree, the following assertions of Borsuk and Brouwer are proved just as before, and the proofs are left as exercises.

Theorem 23.20. (1) *There is no retraction from D^{n+1} onto S^n.*

(2) *Any continuous mapping from a closed disk D^n to itself must have a fixed point.*

Exercise 23.21. Generalize the results of Exercises 4.9–4.17.

Problem 23.22. Show that the degree of the antipodal map from S^n to S^n is 1 if n is odd and -1 if n is even. In particular, the antipodal map is not homotopic to the identity if n is even.

Problem 23.23. (a) Show that no even-dimensional sphere can have a nowhere vanishing vector field. (b) Construct on every odd-dimensional sphere a nowhere vanishing vector field.

To extend the results about antipodal mappings, we need the following generalization of Borsuk's Lemma 4.20. As there, we denote by $P^* = -P$ the antipode of a point P in S^n.

Theorem 23.24. *Let $f\colon S^n \to S^n$ be a continuous map.*

(a) If $f(P^*) = f(P)^*$ for all P in S^n, then the degree of f is odd.
(b) If $f(P^*) = f(P)$ for all P in S^n, then the degree of f is even.

The proof of this requires some new ideas, and is postponed to Appendix E. Assuming this theorem, we can draw the expected consequences:

Corollary 23.25. *(a) If $m < n$, there is no continuous mapping $f\colon S^n \to S^m$ such that $f(P^*) = f(P)^*$ for all P in S^n.*

(b) Any continuous mapping $f\colon S^n \to \mathbb{R}^n$ must map some pair of antipodal points to the same point.

(c) An open set in $\mathbb{R}^n$ cannot be homeomorphic to an open set in $\mathbb{R}^m$ if $n \neq m$.

(d) It is impossible to cover S^n with $n+1$ closed sets, none of which contains a pair of antipodal points.

Exercise 23.26. Prove this corollary.

Problem 23.27. Let $f\colon S^n \to S^n$ be continuous. (a) If $f(P^*) \neq f(P)$ for all P, show that $\deg(f)$ is odd. (b) If $f(P^*) \neq f(P)^*$ for all P, show that $\deg(f)$ is even. (c) Show that the only nontrivial group that can act freely on an even-dimensional sphere is the group with two elements.

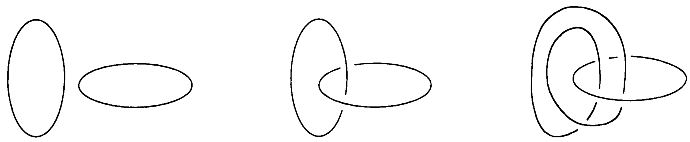
Exercise 23.28. Show that if $n > 1$ any continuous mapping from S^n to $\mathbb{R}P^n$ must map some pair of antipodal points to the same point.

Exercise 23.29. Prove that if $n+1$ bounded measurable objects are given in $\mathbb{R}^n$, then there is a hyperplane that cuts each in half.

Exercise 23.30. (a) State and prove n-dimensional analogues of Exercises 4.24–4.31 and 4.34–4.39. (b) Define the index of a vector field on an open set in $\mathbb{R}^n$ at an isolated singular point, and state and prove the n-dimensional analogues of Proposition 7.5 and its corollaries.

Exercise 23.31. Use Mayer–Vietoris to compute the homology groups of a torus $S^1 \times S^1$, or of any product $S^m \times S^n$.

Homology can be used to define a notion of degree in many other contexts. Here is an important illustration. A *link* in $\mathbb{R}^3$ is a disjoint union of knots. Equivalence is defined just as for knots. A link can be nontrivial even if all the knots occurring in it are trivial. There is a *linking number* that measures how many times two knots intertwine with each other.



0 ±1 ±2

Suppose K and L are disjoint knots. Define a mapping

$$F\colon K \times L \to S^2, \qquad x \times y \mapsto \frac{x-y}{\|x-y\|},$$

which assigns to the pair (x, y) the direction from x to y. This mapping F determines a homomorphism $F_*\colon H_2(K \times L) \to H_2(S^2)$. Both of these homology groups are isomorphic with $\mathbb{Z}$. Choosing orientations of each identifies them with $\mathbb{Z}$, and then F_* is multiplication by some integer, which is defined to be the linking number $l(K, L)$. If a standard orientation is fixed for S^2, the sign of $l(K, L)$ depends on orientations chosen for K and L; it changes sign if either of these orientations are changed. The linking number also changes sign if the roles of K and L are reversed. Note that if K and L are far apart, then F will not be surjective, so this linking number is zero.

Exercise 23.32. (a) Show that the linking number of the two circles in Exercise 22.7 is ± 1, and therefore the linking number of two fibers of the Hopf mapping of Exercise 22.17 is ± 1. (b) Show that the intersection of a small sphere S^3 around a singularity of a plane curve that is a node (see §20c) is two circles whose linking number is ± 1.

It is a fact (see §24c) that the top homology group H_nX of an oriented n-manifold X is $\mathbb{Z}$ (the orientation determining a choice of generator). It follows that any continuous map $f\colon X \to Y$ between oriented n-manifolds has a degree.

Problem 23.33. (a) Show that if X is a compact oriented surface, then $H_2X \cong \mathbb{Z}$. (b) If X is a compact nonorientable surface, show that $H_2X = 0$. (c) If $f: X \to Y$ is a nonconstant analytic map between compact Riemann surfaces, show that the degree defined by homology is the same as the number of sheets of the corresponding branched covering.

23d. Generalized Jordan Curve Theorem

There is a vast generalization of the Jordan curve theorem to higher dimensions. This can be stated as follows:

Theorem 23.34. *If $X \subset S^n$ is homeomorphic to a sphere S^m, then $m \leq n$, and $m < n$ unless $X = S^n$. If $m < n$, the homology groups of the complement are*

$$H_k(S^n \setminus X) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } m = n-1 \text{ and } k = 0, \\ \mathbb{Z} & \text{if } m < n-1 \text{ and } k = 0 \text{ or } k = n-1-m, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the complement has two components if $m = n-1$, and one if $m < n-1$.

The essential point of this theorem is the assertion that the homology of the complement is the same for all embeddings of S^m in S^n. The proof follows the pattern for the Jordan Curve Theorem in the plane very closely, using the full Mayer–Vietoris theorem. We discuss only the new features, leaving details to the reader. First we have the analogous result for embeddings of cubes in S^n.

Proposition 23.35. *If $X \subset S^n$ is homeomorphic to I^m, then $S^n \setminus X$ is connected and $H_k(S^n \setminus X) = 0$ for all $k > 0$.*

Proof. This is by induction on m, the case $m = 0$ being clear since the complement of a point is homeomorphic to $\mathbb{R}^n$, so contractible. For $m > 0$ write I^m as the union of two halves whose intersection is homeomorphic to I^{m-1}; this makes X a union of two subspaces A and B. Applying Mayer–Vietoris to $U = S^n \setminus A$ and $V = S^n \setminus B$, and knowing about $U \cup V$ by induction, we have

$$0 = H_{k+1}(U \cup V) \to H_k(U \cap V) \to H_kU \oplus H_kV.$$

From this it follows that if z is a k-cycle on $S^n \setminus X$ that is not a boundary, then it is not a boundary on $S^n \setminus A$ or $S^n \setminus B$. Continuing to cut the cubes in half, passing to the limit as in Chapter 5, we find that z is not a boundary on $S^n \setminus \{x\}$ for x a point, from which the conclusion follows easily. □

To prove the theorem, also by induction on m, write X as the union of two closed sets A and B homeomorphic to the upper and lower hemispheres of the sphere S^m. Each of A and B is homeomorphic to I^m, and $A \cap B$ is homeomorphic to S^{m-1}. Applying Mayer–Vietoris and the proposition to $U = S^n \setminus A$ and $V = S^n \setminus B$, we get

$$0 \to H_{k+1}(S^n \setminus A \cap B) \to H_k(S^n \setminus X) \to 0$$

if $k > 0$, and

$$0 \to H_1(S^n \setminus A \cap B) \to H_0(S^n \setminus X) \to H_0U \oplus H_0V \to H_0(S^n \setminus A \cap B) \to 0.$$

We know about $S^n \setminus A \cap B$ by induction, so the first display computes $H_k(S^n \setminus X)$ for $k > 0$. If $m < n-1$, $H_1(S^n \setminus A \cap B) = 0$, and the second gives

$$0 \to H_0(S^n \setminus X) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0,$$

from which it follows easily that $H_0(S^n \setminus X) \cong \mathbb{Z}$. If $m = n-1$, then $H_1(S^n \setminus A \cap B) \cong \mathbb{Z}$, and from

$$0 \to \mathbb{Z} \to H_0(S^n \setminus X) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$$

we find similarly that $H_0(S^n \setminus X) \cong \mathbb{Z} \oplus \mathbb{Z}$. If $m = n$, from

$$0 \to H_0(S^n \setminus X) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to 0$$

we see that $H_0(S^n \setminus X) = 0$, so $X = S^n$. From this it follows that no larger sphere can be embedded in S^n, since an n-dimensional subsphere would already map to the whole S^n. □

Exercise 23.36. State and prove analogous results for $X \subset \mathbb{R}^n$ homeomorphic to a sphere.

Exercise 23.37. If $F: D^n \to \mathbb{R}^n$ is continuous and one-to-one, show that $\mathbb{R}^n \setminus F(S^{n-1})$ has two connected components, one the image of the interior of D^n, the other the complement of $F(D^n)$. Deduce the *invariance of domain*: if U is open in $\mathbb{R}^n$, and $F: U \to \mathbb{R}^n$ is a one-to-one continuous mapping, then $F(U)$ is open. Prove the *invariance of dimension*: open sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ cannot be homeomorphic if $n \neq m$.

It is also true that if $F: S^n \to \mathbb{R}^{n+1}$ is an embedding, then its engulfing number around points in the two components of the complements is ± 1 for the bounded component and 0 for the unbounded component (see Proposition 5.20), but this requires more machinery.

Exercise 23.38. Suppose A and B are disjoint closed subsets of S^n, $n > 1$, and two points are given in the complement of $A \cup B$. Show that if A and B do not separate these points, neither does $A \cup B$.

The Mayer–Vietoris sequence can also be used to show that a compact nonorientable surface cannot be topologically embedded in $\mathbb{R}^3$. We sketch a proof in the following problems:

Problem 23.39. (a) Show that if $X \subset \mathbb{R}^3$ is homeomorphic to a Moebius band, then $H_k(\mathbb{R}^3 \setminus X) \cong \mathbb{Z}$ if $k = 0$ or 1, and $H_k(\mathbb{R}^3 \setminus X) = 0$ otherwise. (b) With X as in (a), if $Y \subset X$ corresponds to the boundary circle, show that the map

$$H_1(\mathbb{R}^3 \setminus X) \to H_1(\mathbb{R}^3 \setminus Y) \cong \mathbb{Z}$$

determined by the inclusion of $\mathbb{R}^3 \setminus X$ in $\mathbb{R}^3 \setminus Y$ takes a generator of the first group to twice a generator of the second.

Suppose a subspace X of $\mathbb{R}^3$ were homeomorphic to the projective plane. Write X as a union of a space A homeomorphic to a Moebius band and B homeomorphic to a disk, with $A \cap B$ homeomorphic to the boundary circle of each. Then with $U = \mathbb{R}^3 \setminus A$, $V = \mathbb{R}^3 \setminus B$, Mayer–Vietoris and Exercise 23.36 give an exact sequence

$$H_1(\mathbb{R}^3 \setminus A) \oplus 0 \to H_1(\mathbb{R}^3 \setminus A \cap B) \to H_0(\mathbb{R}^3 \setminus X).$$

By the preceding problem, the image of a generator of $H_1(\mathbb{R}^3 \setminus A \cap B)$ maps to an element α in $H_0(\mathbb{R}^3 \setminus X)$ that is nonzero, but $2 \cdot \alpha = 0$. However, we know that the 0th homology group of any space is a free abelian group, which has no such element.

Problem 23.40. Show similarly that none of the nonorientable compact surfaces can be embedded in $\mathbb{R}^3$.

The following problem generalizes two of the main results of Chapters 6 and 9:

Problem 23.41. Let U be an open subset of $\mathbb{R}^n$. (a) Show that two classes in $H_{n-1}U$ are equal if and only if they map to equal classes in $H_{n-1}(\mathbb{R}^n \setminus \{P\})$ for all P not in U. (b) Show that if the complement of U in $S^n = \mathbb{R}^n \cup \{\infty\}$ is a disjoint union of $m + 1$ compact connected sets, then $H_{n-1}U$ is a free abelian group on m generators.

CHAPTER 24

Duality

24a. Two Lemmas from Homological Algebra

We frequently want to compare different homology and cohomology groups, when we have exact Mayer–Vietoris sequences for each, and maps between them. Assuming that most of the maps are isomorphisms, we want to deduce that the others are as well. There is a general algebraic fact that can be used for this:

Lemma 24.1 (Five-Lemma). *Given a commutative diagram*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

of abelian groups, such that the rows are exact sequences, and all the vertical maps but the middle one are isomorphisms, then the middle map from C to C' must also be an isomorphism.

The proof is by a "diagram chase," which is much easier and enjoyable to do for oneself than to follow when someone else does it. Here is how to show that the map is one-to-one. If c in C maps to 0 in C', then its image in D maps to 0 in D' (by commutativity of the diagram), so c maps to 0 in D (since $D \to D'$ is injective), so c comes from some element b in B (by exactness of the top row). The element b maps to an element b' in B' that maps to 0 in C' (why?), that therefore comes from an element a' in A'. This element a' comes from some element a in A, and this element a must map to b since they have the same images in B'. Since a maps to 0 in C, and b maps to c, c must be 0.

Exercise 24.2. (a) Prove similarly that $C \rightarrow C'$ is surjective. (b) Show that the five-lemma is also valid under the following weaker assumptions, still assuming the rows are exact: (i) each square either commutes or commutes up to sign, i.e., the composite going around one way is plus or minus the composite going around the other way; and (ii) the maps $B \rightarrow B'$ and $D \rightarrow D'$ are isomorphisms, $A \rightarrow A'$ is surjective, and $E \rightarrow E'$ is injective.

If you like this diagram chasing, there is a general process that constructs long exact sequences, which can be used to construct all the Mayer–Vietoris sequences we have seen. For this, one has a commutative diagram of abelian groups

$$\begin{array}{ccccccccc}
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C_{k+1}' & \longrightarrow & C_{k+1} & \longrightarrow & C_{k+1}'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C_k' & \longrightarrow & C_k & \longrightarrow & C_k'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C_{k-1}' & \longrightarrow & C_{k-1} & \longrightarrow & C_{k-1}'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C_{k-2}' & \longrightarrow & C_{k-2} & \longrightarrow & C_{k-2}'' & \longrightarrow & 0, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow
\end{array}$$

where the rows are exact, and the composite of any two successive maps in the columns is zero. One says that the columns are *chain complexes*. The diagram is abridged to saying one has a *short exact sequence of chain complexes*

$$0 \rightarrow C_*' \rightarrow C_* \rightarrow C_*'' \rightarrow 0.$$

For each chain complex (column) one can form *homology groups*. For the center column,

$$H_k(C_*) = Z_k(C_*)/B_k(C_*),$$

where $Z_k(C_*) = \text{Kernel}(C_k \to C_{k-1})$ are the *k-cycles*, and $B_k(C_*) = \text{Image}(C_{k+1} \to C_k)$ are the *k-boundaries*. Similarly for the other two columns. There are maps from $H_k(C_*')$ to $H_k(C_*)$ and from $H_k(C_*)$ to $H_k(C_*'')$, determined by the horizontal maps in the diagram. For example, the map from C_k' maps $Z_k(C_*')$ to $Z_k(C_*)$ and $B_k(C_*')$ to $B_k(C_*)$, so it determines a homomorphism on the quotient group. The interesting maps are *boundary homomorphisms*

$$\partial \colon H_k(C_*'') \to H_{k-1}(C_*').$$

To define these, take a representative z'' in $Z_k(C_*'')$ of a class in $H_k(C_*'')$. Choose an element c in C_k that maps onto z''. Let $\bar{c}$ be the image of c in C_{k-1}. Then $\bar{c}$ maps to 0 in C_{k-1}'', since it has the same image there as z'' does, and z'' is a cycle. So $\bar{c}$ comes from an element c' in C_{k-1}'.

Exercise 24.3. Show that this element c' is a $(k-1)$-cycle, and its homology class in $H_{k-1}(C_*')$ is independent of choices of the representative z'' and the element c that maps onto z''.

The homology class of c' is defined to be the boundary of the homology class of z'': $\partial([z'']) = [c']$. One checks easily that ∂ is a homomorphism of abelian groups.

Proposition 24.4. *The resulting sequence*

$$\ldots \to H_{k+1}(C_*'') \to H_k(C_*') \to H_k(C_*) \to H_k(C_*'') \to H_{k-1}(C_*'') \to \ldots$$

is exact.

The proof is some more diagram chasing, which again we leave as an exercise. There is a similar result when the vertical maps in the diagram go up rather than down. Usually then the indexing is by upper indices, so we have "cochain complexes"

$$C^*: \qquad \ldots \to C^{k-1} \to C^k \to C^{k+1} \to \ldots\ .$$

A short exact sequence $0 \to C^{*\prime} \to C^* \to C^{*\prime\prime} \to 0$ determines a long exact sequence of their *cohomology groups*

$$\ldots \to H^{k-1}(C^{*\prime\prime}) \to H^k(C^{*\prime}) \to H^k(C^*) \\ \to H^k(C^{*\prime\prime}) \to H^{k+1}(C^{*\prime}) \to \ldots\ .$$

Let us see how some of the Mayer–Vietoris sequences we have seen earlier fall out of this formalism. For example, if X is a $\mathscr{C}^\infty$ manifold, and C^kX denotes the vector space of $\mathscr{C}^\infty$ k-forms on X, and

U and V are open sets in X, there is an exact sequence

$$0 \to C^*(U \cup V) \to C^*(U) \oplus C^*(V) \to C^*(U \cap V) \to 0$$

of cochain complexes. The first map takes a form ω on $U \cup V$ to the pair $(\omega|_U, \omega|_V)$, and the second takes a pair (ω_1, ω_2) to the difference $\omega_1|_{U\cap V} - \omega_2|_{U\cap V}$. The exactness of this sequence is clear from the definitions except for the surjectivity of the second map, and that was proved using a partition of unity in Chapter 10. The Mayer–Vietoris sequence then results immediately from Proposition 24.4.

Similarly, for cohomology with compact support, one has a short exact sequence of cochain complexes

$$0 \to C_c^*(U \cap V) \to C_c^*(U) \oplus C_c^*(V) \to C_c^*(U \cup V) \to 0,$$

where the first map takes a form ω on $U \cap V$ to the pair $(\omega^U, -\omega^V)$, where ω^U denotes the extension by 0 from $U \cap V$ to U, and similarly for V; the second map takes (ω_1, ω_2) to $\omega_1{}^{U\cup V} + \omega_2{}^{U\cup V}$. Again, exactness follows from a partition of unity argument, and the Mayer–Vietoris exact sequence results.

For homology, if a space X is a union of open sets U and V, let $C_k(X)^{\mathcal{U}}$ denote the k-chains that are small with respect to the covering $\mathcal{U} = \{U, V\}$ of X. Then there is an exact sequence

$$0 \to C_*(U \cap V) \to C_*(U) \oplus C_*(V) \to C_*(X)^{\mathcal{U}} \to 0$$

of chain complexes, the first taking a chain on $U \cap V$ to the pair consisting of its images on U and on V, and the second taking a pair to the difference of their images on X. The exactness is immediate, giving an exact sequence

$$\begin{aligned} \ldots \to H_{k+1}(C_*(X)^{\mathcal{U}}) \to H_k(U \cap V) \to H_k(U) \oplus H_k(V) \\ \to H_k(C_*(X)^{\mathcal{U}}) \to H_{k-1}(U \cap V) \to \ldots \ . \end{aligned}$$

To complete the proof, one appeals to Proposition 23.12, which says that $H_k(C_*(X)^{\mathcal{U}}) \cong H_k(C_*(X)) = H_k(X)$.

Exercise 24.5. If $0 \to C' \to C \to C'' \to 0$ is an exact sequence of free abelian groups, and G is any abelian group, show that

$$0 \to \mathrm{Hom}(C'', G) \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(C', G) \to 0$$

is also an exact sequence. If $0 \to C_*' \to C_* \to C_*'' \to 0$ is an exact sequence of complexes of free abelian groups, this gives an exact sequence $0 \to \mathrm{Hom}(C_*'', G) \to \mathrm{Hom}(C_*, G) \to \mathrm{Hom}(C_*', G) \to 0$ of cochain complexes, and hence an exact sequence of cohomology groups.

Exercise 24.6. Let C_* and C_*' be chain complexes with boundary maps denoted $\partial_k\colon C_k \to C_{k-1}$ and $\partial_k'\colon C_k' \to C_{k-1}'$, respectively. Define a *map of chain complexes* $f_*\colon C_* \to C_*'$ to be a collection of homomorphisms $f_k\colon C_k \to C_k'$ that commute with the boundary maps. Show that such maps f_* determines homomorphisms from $H_k(C_*)$ to $H_k(C_*')$. Call two maps f_* and g_* *chain homotopic* if there is a collection of maps, $H_k\colon C_k \to C_{k+1}'$ such that

$$g_k - f_k = \partial_{k+1} \circ H_k + H_{k-1} \circ \partial_k$$

for all k. Show that f_* and g_* then determine the same maps from $H_k(C_*)$ to $H_k(C_*')$ for all k.

If $C_*' \to C_*$ is a map of chain complexes such that each $C_k' \to C_k$ is one-to-one, one can define C_k'' to be C_k/C_k', getting an exact sequence $0 \to C_*' \to C_* \to C_*/C_*' \to 0$, so a long exact homology sequence. For example, if Y is a subspace of a topological space X, then $C_*Y \to C_*X$ is one-to-one, so one can define a quotient chain complex C_*X/C_*Y. The homology groups of this complex are denoted $H_k(X, Y)$, and are called the *relative* homology groups. They fit in a long exact sequence

$$\ldots \to H_{k+1}(X, Y) \to H_k(Y) \to H_k(X) \to H_k(X, Y) \to H_{k-1}(Y) \to \ldots .$$

For suitably nice spaces, these relative groups are isomorphic to the homology groups of the space obtained by collapsing (identifying) Y to a point. In many treatments of algebraic topology, these relative groups, and the above sequence, are used for calculation in most situations where we have used the Mayer–Vietoris sequence.

24b. Homology and De Rham Cohomology

In this section we want to prove that the De Rham cohomology groups H^kX of a manifold are dual to the homology groups H_kX, i.e., we want to construct an isomorphism

$$H^kX \xrightarrow{\cong} \operatorname{Hom}(H_kX, \mathbb{R}),$$

generalizing what we did for surfaces for $k = 1$. The idea is similar: one wants to integrate k-forms over k-cubes. This makes sense for differentiable k-cubes, but there is a problem of how to define this for continuous k-cubes that are not differentiable—a problem that we avoided for $k = 1$ by cutting up the path, locally writing the k-form as the differential of a $(k-1)$-form μ, and evaluating μ over the end-points of the subdivided path. For $k>1$, unless the restriction of the k-cube to its boundary is differentiable, this will not work. A more systematic procedure, that does work, is to show that the homology H_kX can be computed by using only differentiable cubes.

A cube $\Gamma: I^k \to X$ is a $\mathscr{C}^\infty$ *cube* if it extends to a $\mathscr{C}^\infty$ mapping on some neighborhood of the cube $I^k \subset \mathbb{R}^k$. Define $C_k^\infty X$ to be the free abelian group on the nondegenerate $\mathscr{C}^\infty$ k-cubes. The boundary ∂ from the k-chains to the $(k-1)$-chains takes $\mathscr{C}^\infty$ cubes to $\mathscr{C}^\infty$ cubes, so we can define the $\mathscr{C}^\infty$ homology groups $H_k^\infty X$ to be the quotient of the closed $\mathscr{C}^\infty$ k-chains modulo the subgroup consisting of boundaries of $\mathscr{C}^\infty$ $(k+1)$-chains. There is an obvious map

$$H_k^\infty X \to H_k X,$$

which, in the language of the preceding section, is given by the map of chain complexes $C_*^\infty X \to C_* X$. We will show that this is an isomorphism.

If ω is a k-form on X, and $\Gamma: I^k \to X$ is a $\mathscr{C}^\infty$ cube, we can define the integral of ω over Γ by

$$\int_\Gamma \omega = \int_{I^k} \Gamma^*(\omega),$$

where $\Gamma^*(\omega)$ is the pull-back form; a form on the cube can be written

$$f(x_1, \ldots, x_k)\, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_k,$$

and the integral of such a form is the usual Riemann integral of the continuous function f on the cube.

Exercise 24.7. Prove "Stokes' theorem" in this context: if ω is a $(k-1)$-form, then

$$\int_\Gamma d\omega = \int_{\partial\Gamma} \omega.$$

From Stokes' theorem, it follows as in the case $k=1$ that there is a map

$$H^k X \to \operatorname{Hom}(H_k^\infty X, \mathbb{R}), \qquad \omega \mapsto \left[\Gamma \mapsto \int_\Gamma \omega\right].$$

We will see that this is also an isomorphism. Combining these two isomorphisms will give the duality we were after.

To prove these isomorphisms, we need to know that the groups $H_k^\infty X$ have many of the same properties as the purely topological groups $H_k X$. For example,

Exercise 24.8. (a) Show that a $\mathscr{C}^\infty$ mapping $f: X \to Y$ of manifolds determines functorial homomorphisms $f_*: H_k^\infty X \to H_k^\infty Y$. (b) If two maps from X to Y are homotopic by a $\mathscr{C}^\infty$ mapping $F: X \times I' \to Y$ (where I' is an open interval containing $[0,1]$), then they determine the same homomorphisms. (c) Deduce that $H_k^\infty U = 0$ for $k > 0$, and $H_0^\infty U = \mathbb{Z}$ if U is a starshaped open set in $\mathbb{R}^n$.

Similarly, one has Mayer–Vietoris exact sequences just as for the groups $H_k X$, and compatible with the maps from groups H_k^∞ to the H_k. In fact, the same construction works in the $\mathscr{C}^\infty$ case, noting that the subdivision operators used to cut cubes into small pieces preserve $\mathscr{C}^\infty$ chains. One modification needs to be made in our proof, however, since the operator A we used in Chapter 23 used a function that is only piecewise differentiable.

Exercise 24.9. Change the function α used in §23b to a $\mathscr{C}^\infty$ function from $[0,1]$ to $[0,1]$ such that $\alpha(0) = 0$ and $\alpha(t) = 1$ if $t \geq 1/2$. With any such α, show that, for any chain Γ, $S \circ A(\Gamma) - \Gamma$ is a boundary. Use this to complete the proof of Mayer–Vietoris for these groups.

To prove these isomorphisms, we need a way to build up arbitrary manifolds out of simple pieces. The following general lemma will suffice for our purposes. Let us call an *open rectangle* in $\mathbb{R}^n$ an open rectangular solid with sides parallel to the axes, i.e., an open set of the form $(a_1, b_1) \times \dots \times (a_n, b_n)$.

Lemma 24.10. *If X is an open set in $\mathbb{R}^n$, then X can be written as the union of two open sets U and V such that each of U and V and $U \cap V$ is a disjoint union of open sets, each of which is a finite union of open rectangles.*

Proof. Take compact sets $K_1 \subset K_2 \subset \dots$ as in the Lemma A.20. Construct a sequence of open sets U_p as follows. Let U_1 be a finite union of rectangles covering K_1, with the closure of each contained in the interior of K_2. Let U_2 be a finite union of rectangles covering $K_2 \setminus \mathrm{Int}(K_1)$, with the closure of each contained in the interior of K_3. Inductively, let U_p be a finite union of rectangles that covers the compact set $K_p \setminus \mathrm{Int}(K_{p-1})$, the closure of each contained in the interior of

K_{p+1} and in the complement of K_{p-2}, and not meeting U_{p-2}. Now let U be the union of the union of all U_p with p even, and let V be the union of the union of all U_p with p odd. □

Lemma 24.11. *If X is a differentiable n-manifold, then X can be written as the union of two open sets U and V such that each of U and V and $U \cap V$ is a disjoint union of open sets, each of which is a finite union of open sets diffeomorphic to open sets in $\mathbb{R}^n$.*

Proof. The argument is the same. Remark A.21 shows that X is a union of compact sets $K_1 \subset K_2 \subset \ldots$ with the same properties. Then the preceding proof, with "rectangle" replaced by "open set diffeomorphic to an open set in $\mathbb{R}^n$" goes over without change. □

Theorem 24.12. *For any manifold X the natural maps $H_k^\infty X \to H_k X$ are isomorphisms.*

Proof. Let us write "$T(X)$" for the statement that the maps from $H_k^\infty X$ to $H_k X$ are isomorphisms for all k. There are three tools:

(1) $T(U)$ is true when U is an open rectangle in $\mathbb{R}^n$.
(2) If U and V are open in a manifold, and if $T(U)$, $T(V)$, and $T(U \cap V)$ are true, then $T(U \cup V)$ is true.
(3) If X is a disjoint union of open manifolds X_α, and each $T(X_\alpha)$ is true, then $T(X)$ is true.

With what we have seen, each of these is easy to prove. (1) follows from the fact that $H_k^\infty U$ and $H_k U$ vanish for $k > 0$, and both are naturally isomorphic to $\mathbb{Z}$ when $k = 0$, cf. Exercise 24.8. (2) follows from the fact that we have Mayer–Vietoris exact sequences for each, with compatible maps between them:

$$\begin{tikzcd}
H_k^\infty U \cap V \arrow[r] \arrow[d] & H_k^\infty U \oplus H_k^\infty V \arrow[r] \arrow[d] & H_k^\infty U \cup V \arrow[r] \arrow[d] & H_{k-1}^\infty U \cap V \arrow[r] \arrow[d] & H_{k-1}^\infty U \oplus H_{k-1}^\infty V \arrow[d] \\
H_k U \cap V \arrow[r] & H_k U \oplus H_k V \arrow[r] & H_k U \cup V \arrow[r] & H_{k-1} U \cap V \arrow[r] & H_{k-1} U \oplus H_{k-1} V
\end{tikzcd}$$

so the five-lemma shows that the middle map is an isomorphism if the others are. (3) follows from the fact that to specify a class of either kind on X is equivalent to specifying a class on each X_α, with all but a finite number of these classes being zero (i.e., $H_k X$ is the direct sum of the groups $H_k X_\alpha$, and similarly for H_k^∞).

We can now use these tools to prove the theorem. We first show that $T(X)$ is true whenever $X \subset \mathbb{R}^n$ is a finite union of open rectangles. This is by induction on the number of rectangles. (1) takes care of one rectangle, and if X is a union of p rectangles, let U be the union of $p-1$ of them and let V be the other. Then $T(U)$ and $T(V)$ are true by induction, and $T(U \cap V)$ is true since $U \cap V$ is also a union of at most $p-1$ rectangles, since the intersection of two rectangles is either empty or a rectangle. Then $T(X)$ is true by (2).

Next we show that $T(X)$ is true whenever X is an open set in $\mathbb{R}^n$. By Lemma 24.10 one can write X as a union of two open sets U and V, such that each of U and V and $U \cap V$ is a disjoint union of open sets, each of which is a finite union of open rectangles. Applying the preceding step and (3) we know that $T(U)$ and $T(V)$ and $T(U \cap V)$ are true, and by (2) again we know that $T(X)$ is true.

Note that since a diffeomorphism between manifolds determines an isomorphism between the corresponding groups, it follows that $T(X)$ is true for any set diffeomorphic to an open set in $\mathbb{R}^n$. The same inductive argument as for rectangles shows that $T(X)$ is true when X is a finite union of open sets, each diffeomorphic to an open set in $\mathbb{R}^n$. For the general case, Lemma 24.11 shows that any manifold X is a union of two open sets U and V such that each of U and V and $U \cap V$ is a disjoint union of open sets, each of which is diffeomorphic to a finite union of open sets in $\mathbb{R}^n$. By the last step and (3) again, $T(U)$ and $T(V)$ and $T(U \cap V)$ are true, and a final application of (2) shows that $T(X)$ is true. □

Theorem 24.13. *For any manifold X the natural maps from H^kX to $Hom(H_k^\infty X, \mathbb{R})$ are isomorphisms.*

Proof. The proof follows exactly the same format, with $T(X)$ being the statement that these maps are isomorphisms for all k. Once (1)–(3) are proved, in fact, the proof is identical. The proof of (1) is the same, and (3) follows from the fact that to specify a class of either kind is equivalent to specifying a class on each X_α (i.e., H^kX is the direct product of the groups H^kX_α, and similarly for $\mathrm{Hom}(H_k^\infty X, \mathbb{R})$). To prove (2), we need to compare the cohomology Mayer–Vietoris sequence with the dual of the Mayer–Vietoris sequence in homology. For brevity write $H_k^\infty X^*$ in place of $\mathrm{Hom}(H_k^\infty X, \mathbb{R})$. We have a diagram

$$\begin{array}{ccccccccc}
H^{k-1}U \oplus H^{k-1}V & \longrightarrow & H^{k-1}U \cap V & \longrightarrow & H^kU \cup V & \longrightarrow & H^kU \oplus H^kV & \longrightarrow & H^kU \cap V \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{k-1}^\infty U^* \oplus H_{k-1}^\infty V^* & \longrightarrow & H_{k-1}^\infty U \cap V^* & \longrightarrow & H_k^\infty U \cup V^* & \longrightarrow & H_k^\infty U^* \oplus H_k^\infty V^* & \longrightarrow & H_k^\infty U \cap V^*.
\end{array}$$

An application of the five-lemma, together with the following exercise, finishes the proof. □

Exercise 24.14. Show that this diagram commutes.

These theorems justify the use of $\mathscr{C}^\infty$ techniques in studying the topology of a differentiable manifold. For example, they show that the De Rham groups depend only on the underlying topology of the manifold. Combining the isomorphisms of the two theorems, one has justified writing $\int_z \omega$ for z a continuous k-cycle and ω a closed $\mathscr{C}^\infty$ k-form on a manifold.

Problem 24.15. Show that for U open in $\mathbb{R}^n$, two classes τ_1 and τ_2 in $H_{n-1}U$ are equal if and only if $\int_{\tau_1} \omega = \int_{\tau_2} \omega$ for all closed $(n-1)$-forms ω on U.

Exercise 24.16. Let X be an n-manifold that can be covered by a finite number of open sets such that any intersection of them is diffeomorphic to a convex open set in $\mathbb{R}^n$. (It is a fact, proved by using a Riemannian metric and geodesics, that any compact manifold has such an open cover.) Show that each H_kX is a finitely generated abelian group, and that each H^kX and H^k_cX is a finite-dimensional vector space.

24c. Cohomology and Cohomology with Compact Supports

In higher dimensions, except in simple cases in the Poincaré lemmas, we have not yet used the higher-dimensional versions of wedging forms that we used on surfaces in Chapter 18. In general the wedge $\omega \wedge \mu$ of a k-form ω and an l-form μ is a $(k+l)$-form. This operation is linear in each factor, and satisfies the identities:

(i) $\mu \wedge \omega = (-1)^{k \cdot l} \omega \wedge \mu$; and
(ii) $d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^k \omega \wedge d\mu$.

Again, we assume these properties from advanced calculus. If either ω or μ has compact support, then $\omega \wedge \mu$ has compact support, since the support of the wedge product is contained in the intersection of the supports of the factors. It follows from (ii) that the wedge product of two closed forms is closed, and that, if one is closed and the other is exact, the wedge product is exact. From this it follows that the wedge product determine products on the cohomology groups

$$\wedge : H^kX \times H^lX \to H^{k+l}X$$

and

$$\wedge : H^k X \times H^l_c X \to H^{k+l}_c X,$$

each by the formula $[\omega] \times [\mu] \mapsto [\omega] \wedge [\mu] = [\omega \wedge \mu]$.

Exercise 24.17. Verify that these are well-defined bilinear mappings. Show that the first satisfies the formula $[\mu] \wedge [\omega] = (-1)^{k \cdot l} [\omega] \wedge [\mu]$. Prove that these products are associative where defined.

Now suppose X is oriented. As we saw in §22e, integrating over the manifold gives a mapping $H^n_c X \to \mathbb{R}$. So we have homomorphisms

$$H^k X \times H^{n-k}_c X \xrightarrow{\wedge} H^n_c X \to \mathbb{R}.$$

This determines linear maps $\mathcal{D}_X \colon H^k X \to \operatorname{Hom}(H^{n-k}_c X, \mathbb{R})$. Explicitly, $\mathcal{D}_X$ takes the class of a closed k-form ω to the homomorphism that takes the class of a closed $(n-k)$-form μ with compact support to the integral $\int_X \omega \wedge \mu$.

Theorem 24.18. *For any oriented manifold X the duality maps*

$$\mathcal{D}_X \colon H^k X \to \operatorname{Hom}(H^{n-k}_c X, \mathbb{R})$$

are isomorphisms.

Proof. The proof is almost identical to that for Theorems 24.12 and 24.13. This time, for (1), note that $H^0 U = \mathbb{R}$ and $H^n_c U \cong \mathbb{R}$ for U an open rectangle, and all other groups vanish, by the Poincaré lemmas; and since $1 \in H^0 U$ maps to the nonzero homomorphism that is integration over U, the map is an isomorphism. For (2), one again has maps from the Mayer–Vietoris sequence for H^k to the dual of the sequence for the H^{n-k}_c. This time the signs involved in the definition mean that the key square in the diagram only commutes up to sign, but that is good enough to apply the five-lemma, cf. Exercise 24.2. We leave the calculation of these signs as an exercise. $\square$

This duality theorem has several corollaries that were not obvious before. For example, the simple fact that $H^0_c X = 0$ whenever X is a connected but not compact manifold implies the

Corollary 24.19. *If X is a connected, oriented, but noncompact n-manifold, then $H^n X = 0$.*

Corollary 24.20. *If X is a connected oriented n-manifold, then the map $H^n_c X \to \mathbb{R}$, $\omega \mapsto \int_X \omega$, is an isomorphism.*

Proof. Since $H^0X = \mathbb{R}$, it follows from the theorem that H^n_cX is one dimensional, and we have seen that the map $H^n_cX \to \mathbb{R}$ is not zero. □

Problem 24.21. Let X be a nonorientable connected n-manifold, and let $p\colon \widetilde{X} \to X$ be the orientation covering of §16a. (a) Construct maps $p^*\colon H^k_cX \to H^k_c\widetilde{X}$ and $p_*\colon H^k_c\widetilde{X} \to H^k_cX$ so that $p_* \circ p^*\mu = 2\cdot\mu$ and $p^* \circ p_*\omega = \omega + \tau_*\omega$, where $\tau\colon \widetilde{X} \to \widetilde{X}$ is the nontrivial deck transformation. (b) Deduce that p^* embeds H^k_cX as the subspace of $H^k_c\widetilde{X}$ consisting of classes of the form $\omega + \tau_*\omega$. (c) Show that $H^n_cX = 0$. In particular, if X is compact, then $H^nX = 0$.

Corollary 24.22 (Poincaré Duality). *If X is a compact oriented n-manifold, then the pairing $H^kX \times H^{n-k}X \to \mathbb{R}$ is a perfect pairing, i.e., for any linear map $\varphi\colon H^{n-k}X \to \mathbb{R}$, there is a unique ω in H^kX such that $\varphi(\mu) = \int_X \omega \wedge \mu$ for all μ in $H^{n-k}X$.*

Problem 24.23. (a) Use this corollary to prove that H^kX is finite dimensional. (b) If $n = 2m$, with m odd, show that the dimension of H^mX is even, and deduce that the Euler characteristic

$$\sum_{k=0}^{n} (-1)^k \dim(H^kX)$$

must be even.

This puts strong restrictions on the homology and cohomology groups of a compact oriented n-manifold. For example, the dimension of H^kX must equal the dimension of $H^{n-k}X$. The skew-commutative algebra structure on the direct sum of the cohomology groups is also useful in many applications.

As we saw for Riemann surfaces these duality theorems can be used to define an *intersection number* $\langle\alpha, \beta\rangle$ for homology classes α in H_pX and β in $H_{n-p}X$, when X is an oriented n-manifold. As in that case, it is possible to do this directly and geometrically, by finding representative cycles that meet transversally, and counting the points of intersection with an appropriate sign. This takes quite a bit of work, however, and one can use duality to define the intersection number quickly: A class α in H_pX determines a linear map from H^pX to $\mathbb{R}$ by $\mu \mapsto \int_\alpha \mu$, and by Poincaré duality there is a unique class ω_α in $H^{n-p}X$ so that $\int_\alpha \mu = \int_X \omega_\alpha \wedge \mu$ for all μ in H^pX. By the same construction,

β in $H_{n-p}X$ determines ω_β in H^pX. So we can define

$$\langle \alpha, \beta \rangle = \int_X \omega_\alpha \wedge \omega_\beta .$$

This is a bilinear pairing, satisfying $\langle \beta, \alpha \rangle = (-1)^{p\cdot(n-p)}\langle \alpha, \beta \rangle$. The fact that $\langle \alpha, \beta \rangle$ is always an integer, however, is not so obvious from this definition, although it can often be verified directly by making constructions for representatives of ω_α and ω_β, as we did for surfaces in Chapter 18.

Exercise 24.24. Suppose X is oriented but not necessarily compact, and X has an open cover as in Exercise 24.16. Construct a homomorphism

$$H_pX \to H_c^{n-p}X, \qquad \alpha \mapsto \omega_\alpha ,$$

characterized by the equality $\int_\alpha \mu = \int_X \omega_\alpha \wedge \mu$ for all μ in H^pX.

Exercise 24.25. Suppose a topological space is a union of an increasing family of open subsets U_i, $U_1 \subset U_2 \subset \ldots$. Show that any element of H_kX is the image of an element of some H_kU_i, and that α_i in H_kU_i and α_j in H_kU_j determine the same element of H_kX if and only if there is some $m \geq \max(i,j)$ such that α_i and α_j have the same image in H_kU_m. This is expressed by saying that H_kX is the *direct limit* of the H_kU_i, and written

$$H_kX = \varinjlim H_kU_i .$$

Exercise 24.26. Suppose a manifold X is an increasing union of open subsets U_i, $U_1 \subset U_2 \subset \ldots$. (a) Use duality to deduce that giving a class η in H^kX is equivalent to giving a collection of classes η_i in H^kU_i for all i such that η_i restricts to η_j if $i > j$. This says that H^kX is the *inverse limit* of the H^kU_i, and is written

$$H^kX = \varprojlim H^kU_i .$$

(Note that this is not obvious from the definition of De Rham groups, even for open sets in $\mathbb{R}^n$.) (b) Show that H_c^kX is the direct limit of the $H_c^kU_i$:

$$H_c^kM = \varinjlim H_c^kU_i .$$

In fact, one can construct cohomology groups $H^k(X;\mathbb{Z})$ for any space X, which are finitely generated abelian groups for manifolds as in Exercise 24.16, and one can find an analogue of the wedge product for these groups; after proving appropriate duality theorems, one has a construction of the intersection pairing whose values are integers. This could be a next chapter, if this book didn't end here. At least now we can give a quick definition of these groups, or of cohomology groups $H^k(X;G)$ with coefficients in any abelian group G, generalizing directly the discussion in §16c. Define a *k-cochain* to be an arbitrary function that assigns to every nondegenerate k-cube in X an element of G; these form a group $C^k(X;G)$. If c is a k-cochain, define the *coboundary* $\delta(c)$ of c to be the $(k+1)$-cochain defined by the formula $\delta(c)(\Gamma) = c(\partial\Gamma)$, where a cochain is extended linearly to be defined on all chains. Then $\delta \circ \delta = 0$, so one can define $H^k(X;G) = Z^k(X;G)/B^k(X;G)$, where $Z^k(X;G)$ is the group of k-cocycles (whose boundary is zero), and $B^k(X;G)$ is the group of k-coboundaries (of $(k-1)$-cochains).

Exercise 24.27. Prove that these groups satisfy the same properties as homology groups, but "dual." For example, maps $f\colon X \to Y$ determine (functorial) homomorphisms $f^*\colon H^kY \to H^kX$, homotopic maps determine the same maps on cohomology groups. State and prove the Mayer–Vietoris theorem for these groups. Construct homomorphisms from $H^k(X;G)$ to $\mathrm{Hom}(H_kX, G)$, and show that these are isomorphisms if $G = \mathbb{R}$.

Project 24.28. If G is an abelian group, and $\mathcal{U}$ is an open covering of a space X, define and study Čech groups $H^k(\mathcal{U};G)$ generalizing the groups $H^1(\mathcal{U};G)$ studied in Chapter 15.

24d. Simplicial Complexes

We have seen the usefulness of triangulating a surface. Many spaces that arise in nature, including many which are not manifolds, admit triangulations. When a space is triangulated, there is a much smaller chain complex that can be used to compute its homology. The general methods of §24a can be used to show that this complex computes the same homology as that using cubical chains.

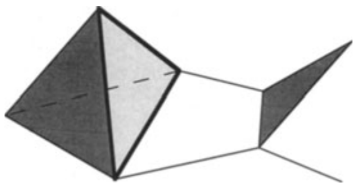
A (finite) *abstract simplicial complex* is a finite set V, called the vertices, and a collection K of subsets of V, called the (abstract) *simplices*, with the property that every subset of a simplex is a simplex. One usually assumes also that every set $\{v\}$ for v in V is a simplex, and one says that K is the simplicial complex. An *n-simplex* is a set σ in K with $n+1$ elements. A subset τ of a simplex σ is called a *face* of σ.

A set of $n+1$ points $P_0, \ldots, P_n$ in a vector space is called *affinely independent* if there is no relation $t_0P_0 + t_1P_1 + \ldots + t_nP_n = 0$ with $t_0, \ldots, t_n$ real numbers satisfying $t_0 + t_1 + \ldots + t_n = 0$ with not all $t_i = 0$. Equivalently, the vectors $P_1 - P_0, P_2 - P_0, \ldots, P_n - P_0$ are linearly independent. In this case the set of points

$$\{t_0P_0 + t_1P_1 + \ldots + t_nP_n:\ t_i \geq 0,\ t_0 + t_1 + \ldots + t_n = 1\}$$

is called the (geometric) *simplex* spanned by the points. It is homeomorphic to an n-dimensional disk.

The *realization* $|K|$ of an abstract simplicial complex K can be constructed by taking the vertices V to be the basis vectors for a vector space, and defining $|K|$ to be the union of the geometric simplices spanned by the abstract simplices in K. In practice one often takes the vertices in a smaller vector space, provided those in any simplex are affinely independent, and two geometric simplices are either disjoint or meet only along common faces.



We want to write down a chain complex for the simplicial complex K. This is simplest if K is *ordered*. This means that a partial ordering is given for the vertices, such that the vertices of each simplex are totally ordered. Each simplex σ then has a unique representation $\sigma = (v_0, \ldots, v_n)$ where the vertices of σ are listed in order. The chain complex C_*K of the ordered simplicial complex K is defined as follows: C_nK is the free abelian group on the n-simplices of K, and the boundary $\partial\colon C_nK \to C_{n-1}K$ is defined by

$$(24.29)\qquad \partial((v_0, \ldots, v_n)) = \sum_{i=0}^{n} (-1)^i (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n).$$

Exercise 24.30. Verify that the composite $\partial \circ \partial\colon C_nK \to C_{n-2}K$ is zero.

The nth homology group $H_n(C_*K)$ of this complex is denoted H_nK.

Exercise 24.31. Suppose K has a vertex v_0 with the property that for every simplex σ in K, the subset consisting of σ and v_0 is also in K;

denote this simplex by (v_0, σ). (Geometrically, $|K|$ is a cone with vertex v_0.) Assume that K is ordered so that v_0 comes before all other vertices. Define maps $H\colon C_nK \to C_{n+1}K$ by the formula

$$H(\sigma) = \begin{cases} (v_0, \sigma) & \text{if } v_0 \text{ is not a vertex of } \sigma, \\ 0 & \text{if } v_0 \text{ is a vertex of } \sigma. \end{cases}$$

Show that $\partial \circ H + H \circ \partial$ is the identity map on C_nK for $n > 0$. Deduce that $H_nK = 0$ for $n > 0$, and that $H_0K \cong \mathbb{Z}$.

A *subcomplex* L of a simplicial complex K is subset of the simplices in K such that whenever a simplex σ is in L, so are all its faces; then L is a simplicial complex, with its vertices a subset of the vertices of V. An ordering of K determines an ordering of L, and one has a canonical map $C_*L \to C_*K$, determining homomorphisms $H_nL \to H_nK$ on homology groups. If L_1 and L_2 are subcomplexes of K, the intersection $L_1 \cap L_2$ and union $L_1 \cup L_2$ are also subcomplexes. These maps determine an exact sequence of chain complexes

$$0 \to C_*(L_1 \cap L_2) \xrightarrow{-} C_*L_1 \oplus C_*L_2 \xrightarrow{+} C_*(L_1 \cup L_2) \to 0,$$

which determines a long exact Mayer–Vietoris sequence

$$\ldots \to H_{n+1}(L_1 \cup L_2) \to H_n(L_1 \cap L_2) \to H_nL_1 \oplus H_nL_2 \\ \to H_n(L_1 \cup L_2) \to \ldots \quad .$$

We want to compare the homology of K with the homology of its geometric realization $|K|$. For each ordered simplex $\sigma = (v_0, \ldots, v_n)$ we need to define a cubical n-chain $\Gamma_\sigma = \Gamma_{(v_0, \ldots, v_n)}$ in $|K|$. If $n = 0$, Γ_σ is the constant 0-chain at v_0. If $n = 1$, Γ_σ is the path from v_0 to v_1: $\Gamma_\sigma(t) = tv_1 + (1-t)v_0$. In general, define $\Gamma_\sigma\colon I^n \to |K|$ inductively by the formula

$$\Gamma_\sigma(t_1, \ldots, t_n) = t_nv_n + (1-t_n)\Gamma_{(v_0, \ldots, v_{n-1})}(t_1, \ldots, t_{n-1}).$$

Writing this out, we have

$$(24.32) \quad \Gamma_\sigma(t_1, \ldots, t_n) = \sum_{k=0}^{n} t_k(1-t_{k+1}) \cdot \ldots \cdot (1-t_n)v_k,$$

where, when $k = 0$, t_0 is set equal to 1.

Proposition 24.33. (*a*) *The map* $\sigma \to \Gamma_\sigma$ *determines a homomorphism* $C_*K \to C_*|K|$ *of chain complexes.* (*b*) *The induced homomorphisms* $H_nK \to H_n|K|$ *are isomorphisms.*

Proof. For (a), we must show that $\partial\Gamma_\sigma = \Sigma_{i=0}^n(-1)^i\Gamma_{(v_0,\ldots,\hat{v}_i,\ldots,v_n)}$, where the ^ denotes an omitted vertex. From the definition of $\partial\Gamma_\sigma$ as $\Sigma_{i=1}^k(-1)^i(\partial_i^0\Gamma_\sigma - \partial_i^1\Gamma_\sigma)$, this follows from the following three calculations, which are simple exercises, using (24.32):

(i) $\partial_1^1\Gamma_\sigma = \Gamma_{(v_1,\ldots,v_n)}$;

(ii) $\partial_i^1\Gamma_\sigma$ is a degenerate $(n-1)$-cube if $i>1$; and

(iii) $\partial_i^0\Gamma_\sigma = \Gamma_{(v_0,\ldots,\hat{v}_i,\ldots,v_n)}$.

The proof of (b) will be by the (by now) familiar induction using Mayer–Vietoris, as follows. If L_1 and L_2 are subcomplexes of K, we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_*(L_1\cap L_2) & \xrightarrow{-} & C_*L_1\oplus C_*L_2 & \xrightarrow{+} & C_*(L_1\cup L_2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_*(|L_1\cap L_2|) & \xrightarrow{-} & C_*|L_1|\oplus C_*|L_2| & \xrightarrow{+} & C_*(|L_1\cup L_2|) & \longrightarrow & 0.
\end{array}$$

This gives a corresponding commutative diagram of long exact sequences, and the five-lemma shows that if (b) is true for L_1 and L_2 and $L_1\cap L_2$, then (b) is also true for $L_1\cup L_2$.

We can now prove (b) by induction on the number of simplices in the simplicial complex K. Take any vertex v of K. Let L_1 be the subcomplex consisting of all simplices of K that are contained in a simplex of K that contains v, and let L_2 be the subcomplex consisting of all simplices of K that do not contain v. Then (b) is known for L_1 by Exercise 24.31, and (b) is known for L_2 and $L_1\cap L_2$ by induction on the number of vertices. The preceding argument then shows that (b) holds for $K = L_1\cup L_2$. □

Corollary 24.34. *If K and L are simplicial complexes whose geometric realizations are homeomorphic, then $H_nK\cong H_nL$ for all n.*

Proof. This follows from the fact that a homeomorphism between spaces induces an isomorphism between their homology groups. □

In the early days of algebraic topology, the homology of a compact space X was defined by triangulating the space, i.e., finding a homeomorphism between some $|K|$ and X, and taking the homology H_*K. With this as the definition the assertion of the preceding corollary—that homology is a topological invariant of the space—was a serious problem.

The preceding discussion depended on a choice of ordering of the simplicial complex, which is how one would usually use the result in calculations. The following exercise shows how this can be circumvented:

Exercise 24.35. For an abstract simplicial complex K, define C_nK to be the quotient of the free abelian group on the set of symbols $(v_0, \ldots, v_n)$, where $v_0, \ldots, v_n$ is an (ordered) set of vertices spanning an n-simplex of K, modulo the subgroup generated by relations

$$(v_0, \ldots, v_n) - \operatorname{sgn}(\tau)(v_{\tau(0)}, \ldots, v_{\tau(n)}),$$

for all permutations τ in the symmetric group ${}_{n+1}$, where $\operatorname{sgn}(\tau) = \pm 1$ is the sign of the permutation. Then C_nK is a free abelian group of rank equal to the number of n-simplices, but with basis elements only specified up to multiplication by ± 1. (a) Show that formula (24.29) determines a boundary map $\partial\colon C_nK \to C_{n-1}K$, with $\partial \circ \partial = 0$. (b) Given an ordered n-simplex $(v_0, \ldots, v_n)$, define $\Gamma'_{(v_0, \ldots, v_n)}$ to be $\Gamma_{(b_0, \ldots, b_n)}$, where b_k is the barycenter of the simplex spanned by the first $k+1$ vertices, i.e., $b_k = 1/(k+1)(v_0 + v_1 + \ldots + v_k)$. Define a map from C_*K to $C_*|K|$ by sending $(v_0, \ldots, v_n)$ to the sum $\Sigma \operatorname{sgn}(\tau)\Gamma'_{(v_{\tau(0)}, \ldots, v_{\tau(n)})}$, the sum over all τ in $\mathfrak{S}_{n+1}$. Show that this determines a homomorphism of chain complexes, and show that the resulting map in homology is an isomorphism. (c) Show that an ordering of K determines an isomorphism of the complex defined earlier with the complex defined in this exercise.

Problem 24.36. If c_i is the number of i-simplices in K, show that the Euler characteristic is the alternating sum of the numbers of simplices:

$$\Sigma(-1)^i c_i \;=\; \Sigma(-1)^i \dim(H_i(|K|)),$$

generalizing what we have seen for surfaces.

Problem 24.37. (a) If $\mathcal{U} = \{U_v,\ v \in V\}$ is a finite collection of open sets whose union is a space X, define a simplicial complex, called the *nerve* of $\mathcal{U}$ and denoted $N(\mathcal{U})$, by taking V to be the vertices, and defining the simplices to be the subsets S such that the intersection of the U_v for v in S is nonempty. Verify that $N(\mathcal{U})$ is a simplicial complex.

(b) If K is any simplicial complex, and v is a vertex in K, define an open set $\mathrm{St}(v)$ in $|K|$, called the *star* of v, to be the union the "interiors" of the simplices that contain v, i.e., $\mathrm{St}(v)$ is the complement in $|K|$ of the union of those $|\sigma|$ for which σ does not contain v. Show that the open sets $\{\mathrm{St}(v), v \in V\}$ form an open covering of $|K|$, and that the nerve of this covering is the same as K.

(c) Suppose $\mathcal{U}$ is an open covering of X as in (a), with the property that for all $v_0, \ldots, v_r$ in V, $U_{v_0} \cap \ldots \cap U_{v_r}$ is connected and $H_k(U_{v_0} \cap \ldots \cap U_{v_r}) = 0$ for all $k > 0$. Construct a homomorphism of chain complexes from $C_*(N(\mathcal{U}))$ to C_*X, and show that it determines an isomorphism from $H_k(N(\mathcal{U}))$ to H_kX for all k.

APPENDICES

These appendices collect some facts used in the text. The beginnings of Appendices A, B, and C state definitions and basic results from point set topology, calculus, and algebra that should be reasonably familiar, together with proofs of a few basic results that may be slightly less so. Each of these appendices ends with some more technical results that may be consulted as the need arises. Appendix D contains two technical lemmas about vector fields in the plane, as well as some basic definitions about coordinate charts and differential forms on surfaces. Appendix E contains a proof of Borsuk's general theorem on antipodal maps that was stated in Chapter 23.

Conventions and Notation

A *closed rectangle* in $\mathbb{R}^2$ has sides parallel to the axes, so is a subset of the form $[a, b] \times [c, d]$, with $a < b$ and $c < d$. An *open rectangle* is a product of two open intervals, usually finite, but we occasionally allow them to be infinite.

The *unit interval* I is $[0, 1] = \{x \in \mathbb{R}: 0 \leq x \leq 1\}$.
The *n-dimensional disk* D^n is

$$D^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n: x_1^2 + \ldots + x_n^2 \leq 1\}.$$

The *n-sphere* S^n is

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}: x_1^2 + \ldots + x_{n+1}^2 = 1\}.$$

The origin $(0, 0, \ldots, 0)$ in $\mathbb{R}^n$ is often denoted simply by 0.

APPENDIX A

Point Set Topology

A1. Some Basic Notions in Topology

A *topology* on a set X is a collection of subsets, called the open sets, including X itself and the empty set, such that any union of open sets is open, and any finite intersection of open sets is open. A *topological space* is a set X together with a topology. A collection of open sets is a *basis* for the open sets if any open set is a union of sets in the basis. For example, if X is a metric space, the open balls $B_\varepsilon(x) = \{y \in X\colon \text{distance}(y, x) < \varepsilon\}$ form a basis for a topology on X. In particular, Euclidean space $\mathbb{R}^n$ with its usual distance function is a topological space. A *neighborhood* of a point in a topological space is an open set containing the point—or, occasionally, any set containing such an open set.

Any subset Y of a topological space is a topological space with the induced topology: the open sets are those of the form $U \cap Y$, for U open in X. Such Y is called a *topological subspace* of X. In particular, any subset of $\mathbb{R}^n$ is a topological space. A subset Y is *closed* if its complement is open. A map $f\colon X \to Y$ from one topological space to another is *continuous* if $f^{-1}(U)$ is open in X for every open set U in Y. A bijection $f\colon X \to Y$ is a *homeomorphism* if f and f^{-1} are continuous.

A topological space X is *Hausdorff* if, for any two distinct points in X, there are disjoint open sets, one containing one of the points, the other containing the other. Any metric space is Hausdorff. Although we seldom need to assume spaces are Hausdorff, the reader will lose little by assuming that all spaces occurring in the book are Hausdorff.

A subset K of a space X is called *compact*, if, for any collection of open sets $\{U_\alpha: \alpha \in \mathscr{A}\}$ such that K is contained in the union of the U_α, there is a finite subset $\{\alpha(1), \ldots, \alpha(m)\}$ of $\mathscr{A}$ so that K is contained in the union $U_{\alpha(1)} \cup \ldots \cup U_{\alpha(m)}$. The following are some basic facts about compact spaces:

(A.1) If $f: X \to Y$ is continuous, and K is a compact subset of X, then $f(K)$ is a compact subset of Y.
(A.2) A compact subset of a Hausdorff space is closed.
(A.3) If $f: X \to Y$ is continuous and bijective, and X is compact and Y is Hausdorff, then f is a homeomorphism.
(A.4) A subset K of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Exercise A.5. If K and L are disjoint compact subsets in a Hausdorff space X, show that there are disjoint open sets in X, one containing K, the other containing L.

Exercise A.6. If K is compact, and, for each positive integer n, A_n is a nonempty subset of K, show that there is a limit point, i.e., a point P in K such that every neighborhood of P meets A_n for an infinite number of integers n.

Exercise A.7. (a) Show that a rectangle $[a, b] \times [c, d]$ is homeomorphic to the closed unit disk $\{(x, y): x^2 + y^2 \leq 1\}$. (b) Show that $\mathbb{R}^2$ is homeomorphic to the open unit disk $\{(x, y): x^2 + y^2 < 1\}$.

A subset X of $\mathbb{R}^n$ is *convex* if, for any points P and Q in X, the line segment $\{t \cdot P + (1 - t) \cdot Q: 0 \leq t \leq 1\}$ from P to Q is contained in X.

Problem A.8. Show that any compact, convex subset of $\mathbb{R}^n$ that contains a nonempty open set is homeomorphic to the closed unit disk

$$D^n = \{(x_1, \ldots, x_n): x_1{}^2 + \ldots + x_n{}^2 \leq 1\}.$$

If ∂K is the *boundary* of K, i.e., ∂K is the set of points of K such that every neighborhood contains points inside and outside K, show that there is a homeomorphism from K to D^n that maps ∂K homeomorphically onto the boundary $S^{n-1} = \partial D^n$.

If X and Y are topological spaces, the Cartesian products $U \times V$ of open sets U in X and V in Y form a basis for a topology in the Cartesian product $X \times Y$, called the *product* topology.

If X and Y are topological spaces, the *disjoint union* $X \sqcup Y$ is a topological space. A set in the disjoint union is open when it is the union of an open set in X and an open set in Y. More generally, if $\{X_\alpha: \alpha \in \mathscr{A}\}$ is any collection of topological spaces, the disjoint union $\bigsqcup X_\alpha$ is a topological space, with open sets disjoint unions of open sets in each X_α.

Any set X can be made into a topological space with the *discrete topology*, in which every subset of X is open. Equivalently, all points are open.

The *interior* of a subset A of a topological space, denoted $\mathrm{Int}(A)$, is the set of points that have a neighborhood contained in A. The *closure* of a subset A, denoted $\overline{A}$, is the intersection of all closed sets containing A.

A2. Connected Components

A topological space X is *connected* if it cannot be written as a union of two nonempty disjoint sets, each of which is open in X.

Exercise A.9. Show that each of the following is equivalent to X being connected: (i) X has no nonempty proper subset that is both open and closed; (ii) X cannot be written as the union of two nonempty disjoint closed subsets; and (iii) there is no continuous mapping from X onto the discrete space $\{0, 1\}$.

The following are basic facts about connected spaces:

(A.10) If $f: X \to Y$ is a continuous, surjective mapping, and X is connected, then Y is connected.

(A.11) If X is a subspace of a space Y, and X is connected, then the closure $\overline{X}$ of X in Y is also connected.

(A.12) If X is a union of a family of subspaces X_α, each of which is connected, and each pair of which have nonempty intersection, then X is connected.

(A.13) The connected subsets of $\mathbb{R}$ are the intervals.

A *connected component* of X is a connected subset that is not contained in any larger connected subset. Each connected component is closed in X. Any two connected components of X are disjoint. The union of all connected subsets of X containing a point x is a connected component, called the *connected component of x in X*. The space X is a disjoint union of its connected components.

Exercise A.14. Let $X \subset \mathbb{R}^2$ be the union of the points $(0,0)$, $(0, 1)$, and the lines $\{1/n\} \times [0, 1]$, $n = 1, 2, \ldots$. Show that these are the connected components of X, but whenever X is written as the union of two open and closed subsets, the points $(0,0)$ and $(0, 1)$ belong to the same subset.

A space is called *locally connected* if for every neighborhood V of every point x, there is a connected open neighborhood U of x that is contained in V.

Exercise A.15. If X is locally connected, show that all the connected components of X are open in X.

A space X is *path-connected* if, for any two points x and y in X, there is a continuous mapping γ from an interval $[a, b]$ to X that maps a to x and b to y. A space is *locally path-connected* if every neighborhood of every point contains a path-connected neighborhood of the point.

Exercise A.16. If X is locally path-connected, show that all the connected components of X are path-connected and open in X.

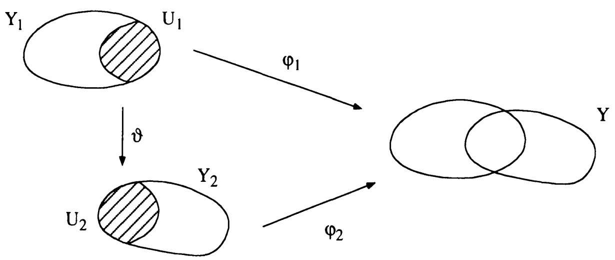
In particular, for U open in the plane or $\mathbb{R}^n$, the connected components of U are open and path-connected.

A3. Patching

If a topological space X is the union of two sets A and B, both open or both closed, and $f: A \to Y$ and $g: B \to Y$ are continuous mappings from A and B to a space Y, such that f and g agree on $A \cap B$, then there is a unique continuous mapping h from X to Y that agrees with f on A and with g on B.

If Y is a topological space, and R is any equivalence relation on Y, the set Y/R of equivalence classes is given the *quotient topology*: a set U is open in Y/R exactly when its inverse image in Y is open. If $f: Y \to Z$ is a continuous mapping that maps all points in each equivalence class to the same point, then f determines a continuous mapping $\bar{f}: Y/R \to Z$ so that the composite $Y \to Y/R \to Z$ is f.

Suppose Y_1 and Y_2 are two spaces, with open subsets U_1 of Y_1 and U_2 of Y_2, and a homeomorphism $\vartheta: U_1 \to U_2$ is given between them. Then one can *patch* (or *glue*, or *clutch*) the spaces Y_1 and Y_2 together, to form a space Y. There will be maps $\varphi_1: Y_1 \to Y$ and $\varphi_2: Y_2 \to Y$; Y will be the union of the open subsets $\varphi_1(Y_1)$ and $\varphi_2(Y_2)$, each φ_i will map Y_i homeomorphically onto $\varphi_i(Y_i)$, with $\varphi_1(U_1) = \varphi_2(U_2)$, and ϑ will be the composite $\varphi_2^{-1} \circ \varphi_1$ on U_1.



One can construct Y as the quotient space $Y_1 \amalg Y_2/R$, where R is the equivalence relation consisting of pairs $(u_1, \vartheta(u_1))$ for u_1 in U_1, and of course the symmetric pairs $(\vartheta(u_1), u_1)$, together with all pairs (y_1, y_1) for y_1 in Y_1 and all pairs (y_2, y_2) for y_2 in Y_2.

More generally, suppose we have a collection Y_α of spaces, for α in an index set $\mathcal{A}$, and, for each α and β in $\mathcal{A}$, we have an open subset $U_{\alpha\beta}$ of Y_α, and a homeomorphism

$$\vartheta_{\beta\alpha}: U_{\alpha\beta} \to U_{\beta\alpha}.$$

These should satisfy the conditions:

(1) $U_{\alpha\alpha} = Y_\alpha$, and $\vartheta_{\alpha\alpha}$ is the identity on Y_α; and
(2) for any α, β, and γ in $\mathcal{A}$, $\vartheta_{\beta\alpha}(U_{\alpha\beta} \cap U_{\alpha\gamma}) \subset U_{\beta\gamma}$ and

$$\vartheta_{\gamma\alpha} = \vartheta_{\gamma\beta} \circ \vartheta_{\beta\alpha} \qquad \text{on} \quad U_{\alpha\beta} \cap U_{\alpha\gamma}.$$

In particular, $\vartheta_{\alpha\beta} \circ \vartheta_{\beta\alpha}$ is the identity on $U_{\alpha\beta}$. Set

$$Y = \coprod_{\alpha \in \mathcal{A}} Y_\alpha / R,$$

where R is the equivalence relation: y_α in Y_α is equivalent to y_β in Y_β if and only if $y_\alpha \in U_{\alpha\beta}$, $y_\beta \in U_{\beta\alpha}$, and $\vartheta_{\beta\alpha}(y_\alpha) = y_\beta$.

Let φ_α be the map from Y_α to Y that takes a point to its equivalence class. Give Y the quotient topology, which means that a set U in Y is open if and only if each $\varphi_\alpha^{-1}(U)$ is open in Y_α.

Lemma A.17. (1) *Each $\varphi_\alpha(Y_\alpha)$ is open in Y;*
(2) *φ_α is a homeomorphism of Y_α onto $\varphi_\alpha(Y_\alpha)$;*
(3) *Y is the union of the sets $\varphi_\alpha(Y_\alpha)$;*
(4) *$\varphi_\alpha(U_{\alpha\beta}) = \varphi_\beta(U_{\beta\alpha})$; and*
(5) *on $U_{\alpha\beta}$, $\vartheta_{\beta\alpha} = \varphi_\beta^{-1} \circ \varphi_\alpha$.*

Proof. The fact that φ_α is one-to-one onto its image, and the assertions (3)–(5), are set-theoretic verifications, and left to the reader. The topology on Y is defined to make each φ_α continuous. To prove (1) and (2), it suffices to verify that if U is open in some Y_α, then $\varphi_\alpha(U)$ is open in Y, i.e., that, for all β, $\varphi_\beta^{-1}(\varphi_\alpha(U))$ is open in Y_β. But $\varphi_\beta^{-1}(\varphi_\alpha(U)) = \vartheta_{\beta\alpha}(U \cap U_{\alpha\beta})$, which is open since $U \cap U_{\alpha\beta}$ is open in $U_{\alpha\beta}$ and $\vartheta_{\beta\alpha}$ is a homeomorphism. □

Exercise A.18. Make a similar construction if each $U_{\alpha\beta}$ is a closed subset of Y_α.

A4. Lebesgue Lemma

We make frequent use of the following lemma:

Lemma A.19. (Lebesgue Lemma). *Given any covering of a compact metric space K by open sets, there is an $\varepsilon > 0$ such that any subset of K of diameter less than ε is contained in some open set in the covering.*

Proof. If not, there is for every integer n a subset A_n of K with diameter less than $1/n$ and not contained in any open set of the covering. From the fact that K is compact it follows that there is a limit point P, see Exercise A.6. Let U be an open set of the covering that contains P, and take $r > 0$ so all points within distance r of P are contained in U. There must be (infinitely many) n with $1/n < r/2$ such that A_n meets the open ball of radius $r/2$ around P. But such A_n must be contained in U, a contradiction. □

The following lemma will be used in Appendix B to construct a partition of unity:

Lemma A.20. *If U is an open set in $\mathbb{R}^n$, there is a sequence of compact subsets $K_1, K_2, \ldots$, whose union is U, and so that*

$$K_1 \subset \mathrm{Int}(K_2) \subset K_2 \subset \mathrm{Int}(K_3) \subset \ldots \subset K_n \subset \mathrm{Int}(K_{n+1}) \subset \ldots .$$

Proof. Start with any countable sequence of open sets U_i that cover U such that the closure $\overline{U}_i$ is compact and contained in U; for example, one can take the U_i to be balls at centers with rational coordinates and rational radii. Take $K_1 = \overline{U}_1$. Then take $K_2 = \overline{U}_1 \cup \ldots \cup \overline{U}_p$, where p is minimal such that K_1 is contained in $U_1 \cup \ldots \cup U_p$, and so on: if $K_m = \overline{U}_1 \cup \ldots \cup \overline{U}_s$, take $K_{m+1} = \overline{U}_1 \cup \ldots \cup \overline{U}_t$ where t is minimal so that K_m is contained in $U_1 \cup \ldots \cup U_t$. □

Remark A.21. The lemma is true, with the same proof, when U is replaced by any manifold whose topology has a countable basis of open sets.

APPENDIX B

Analysis

B1. Results from Plane Calculus

We list the basic results from calculus that were used in Chapters 1 and 2. As in those chapters, for simplicity, differentiable functions on a closed interval or rectangle are assumed to have differentiable extensions to some open neighborhood. Integrals of continuous functions on a closed interval, or a closed rectangle, are defined as limits of Riemann sums. The next five basic facts from calculus are stated for easy reference, in the form we need. Consult your favorite calculus book for proofs.

(B.1) Fundamental Theorem of Calculus. *If a continuous function f is the derivative of a function F on an interval $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

(B.2) Mean Value Theorem. *If f is continuous on an interval $[a, b]$, there is an x^* with $a < x^* < b$ such that*

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(x^*).$$

(B.3) Chain Rule. *If $\gamma(t) = (x(t), y(t))$, $a \le t \le b$ is a differentiable path on an interval $[a, b]$, and f is a differentiable function on a neighborhood of*

$\gamma([a,b])$, then $f\circ\gamma$ is differentiable on $[a,b]$, and

$$\frac{d}{dt}(f(\gamma(t))) = \frac{\partial f}{\partial x}(x(t),y(t))\frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t),y(t))\frac{dy}{dt}.$$

(B.4) Equality of Mixed Partial Derivatives. *If f is a $\mathscr{C}^\infty$ function on an open set in the plane, then*

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

(B.5) Fubini's Theorem. *If f is a continuous function on a rectangle $R=[a,b]\times[c,d]$, then*

$$\iint_R f(x,y)\,dx\,dy = \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx = \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy.$$

Proposition B.6 (Green's Theorem for a Rectangle). *If p and q are continuously differentiable functions on a rectangle $R=[a,b]\times[c,d]$, then*

$$\iint_R\left(\frac{\partial q}{\partial x}-\frac{\partial p}{\partial y}\right)dx\,dy = \int_a^b p(x,c)\,dx + \int_c^d q(b,y)\,dy - \int_a^b p(x,d)\,dx - \int_c^d q(a,y)\,dy.$$

Proof. By Fubini's theorem and the fundamental theorem of calculus,

$$\iint_R \frac{\partial q}{\partial x}dx\,dy = \int_c^d\left[\int_a^b \frac{\partial q}{\partial x}dx\right]dy = \int_c^d [q(b,y)-q(a,y)]\,dy;$$

$$\iint_R \frac{\partial p}{\partial y}dx\,dy = \int_a^b\left[\int_c^d \frac{\partial q}{\partial y}dy\right]dx = \int_a^b [p(x,d)-p(x,c)]\,dx.$$

Green's theorem results by subtracting these two equations. □

Writing $\omega = p(x,y)\,dx + q(x,y)\,dy$, this says that

$$\iint_R d\omega = \int_{\gamma_1}\omega + \int_{\gamma_2}\omega - \int_{\gamma_3}\omega - \int_{\gamma_4}\omega,$$

where γ_1, γ_2, γ_3, and γ_4 are the four sides of the rectangles, as in Chapter 1.

Corollary B.7. *If $d\omega = 0$, then*

$$\int_{\gamma_1}\omega + \int_{\gamma_2}\omega = \int_{\gamma_3}\omega + \int_{\gamma_4}\omega.$$

Exercise B.8. If f is continuous on $[a,b]$, and $|f(t)| \le M$ on $[a,b]$, show that $|\int_a^b f(t)\,dt| \le M \cdot (b-a)$.

The following result will be used in Appendix D:

Lemma B.9. *If f is a $\mathscr{C}^\infty$ function in a neighborhood of $P=(a,b)$ in $\mathbb{R}^2$, with $f(P)=0$, then there are $\mathscr{C}^\infty$ functions f_1 and f_2 so that*

$$f(x,y) \;=\; (x-a)f_1(x,y) + (y-b)f_2(x,y)$$

in a neighborhood of P.

Proof. We may assume $P=(0,0)$. By the fundamental theorem of calculus and the chain rule,

$$\begin{aligned} f(x,y) &= \int_0^1 \frac{\partial}{\partial t}(f(tx,ty))\,dt \\ &= x\int_0^1 \frac{\partial f}{\partial x}(tx,ty)\,dt + y\int_0^1 \frac{\partial f}{\partial y}(tx,ty)\,dt. \end{aligned}$$

The functions

$$f_1(x,y) \;=\; \int_0^1 \frac{\partial f}{\partial x}(tx,ty)\,dt \quad\text{and}\quad f_2(x,y) \;=\; \int_0^1 \frac{\partial f}{\partial y}(tx,ty)\,dt$$

are the required $\mathscr{C}^\infty$ functions. □

B2. Partition of Unity

For construction of the Mayer–Vietoris sequence for open sets in the plane, we need the following result:

Proposition B.10 (Partition of Unity). *Suppose an open set U in $\mathbb{R}^n$ is the union of a sequence $U_1, U_2, \ldots$ of open sets with the property that each point is contained in U_i for only finitely many i. Then there is a sequence of nonnegative $\mathscr{C}^\infty$ functions φ_i on U such that the closure (in U) of the support of φ_i is contained in U_i, and $\Sigma_{i=1}^\infty \varphi_i \equiv 1$ on U.*

(We will construct these functions so that only finitely many φ_i are nonzero in a neighborhood of any point of U, so the sum is a well-defined $\mathscr{C}^\infty$ function.)

Proof. There are several steps.

Step 1. There is a $\mathscr{C}^\infty$ function f on $\mathbb{R}$ that is zero on the negative half line and positive on the positive half line. Such a function is

$$f(x) = \begin{cases} \exp\left(-\dfrac{1}{x}\right) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Exercise B.11. Verify that this is infinitely differentiable by showing that any derivative of $\exp(-1/x)$ has the form $(P(x)/x^m)\exp(-1/x)$ for some polynomial P and some exponent m.

Step 2. Given a bounded rectangle $(a_1, b_1) \times \ldots \times (a_n, b_n)$, there is a $\mathscr{C}^\infty$ function h in $\mathbb{R}^n$ that is positive on the rectangle, and zero outside the rectangle. In fact, the function $g(x) = f(x) \cdot f(1-x)$ is positive on $(0, 1)$ and zero outside this interval, and this leads to the required function

$$h(x_1, \ldots, x_n) = \prod_{i=1}^{n} g\left(\frac{x_i - a_i}{b_i - a_i}\right).$$

Step 3. If A is a compact subset of U, there are $\mathscr{C}^\infty$ functions σ_i so that the closure of the support of σ_i is contained in U_i, and the sum $\Sigma_{i=1}^{\infty} \sigma_i$ is everywhere positive on A. To construct them cover A by a finite number of rectangles R_α such that the closure of each R_α is contained in some U_i, and use Step 2 to construct h_α that are positive on R_α and zero outside. Take σ_1 to be the sum of those h_α such that $\bar{R}_\alpha$ is contained in U_1, let σ_2 be the sum of those among the others such that the closure of the support is contained in U_2, and continue in this way until all h_α are used; set the other σ_i equal to zero.

Step 4. To complete the proof, write U as an increasing union of compact sets $K_1 \subset K_2 \subset \ldots$ as in Lemma A.20. Let $A_j = K_j \setminus \mathrm{Int}(K_{j-1})$ (where we set $K_j = \emptyset$ for $j \leq 0$). Let $W_j = \mathrm{Int}(K_{j+1}) \setminus K_{j-2}$. Note that A_j is a compact subset of the open set W_j. Apply Step 3 to each compact set $A_j \subset W_j$ with its open covering $\{U_i \cap W_j\}$, obtaining functions σ_{ij} so that the closure of the support of σ_{ij} is contained in $U_i \cap W_j$ and with $\Sigma_{i=1}^{\infty} \sigma_{ij}$ everywhere positive on A_j. Define ψ_i to be the sum $\Sigma_{i=1}^{\infty} \sigma_{ij}$. Only finitely many (at most three) terms in this sum are nonzero in some neighborhood of any point, so ψ_i is a $\mathscr{C}^\infty$ function the closure of whose support is contained in U_i. The sum $\psi = \Sigma_{i=1}^{\infty} \psi_i$ is similarly a $\mathscr{C}^\infty$ function, and ψ is positive on each A_j, so it is positive on all of U. Now

$$\varphi_i = \frac{\psi_i}{\psi}$$

satisfies all the required conditions. □

Exercise B.12 (Partition of Unity for Arbitrary Coverings). Suppose

$\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ is an arbitrary open covering of an open set U in $\mathbb{R}^n$. Show that there is a sequence of nonnegative $\mathscr{C}^\infty$ functions $\varphi_1, \varphi_2, \ldots$ on U such that: (i) the closure (in U) of the support of φ_i is contained in some open set $U_{\alpha(i)}$; (ii) for each P in U there is a neighborhood of P such that only finitely many φ_i are nonzero on the neighborhood; and (iii) $\sum_{i=1}^{\infty} \varphi_i \equiv 1$.

Exercise B.13. Extend these results on partitions of unity to the case where U is any manifold that has a countable basis of open sets.

Exercise B.14. For $0 < r_1 < r_2$ construct a $\mathscr{C}^\infty$ function ψ on the plane that vanishes on the disk of radius r_1 centered at the origin, and is identically 1 outside the disk of radius r_2 centered at the origin, and takes values in the interval $(0, 1)$ between the two circles.

APPENDIX C

Algebra

C1. Linear Algebra

In this book, unless otherwise stated, vector spaces are real vector spaces. The vector space $\mathbb{R}^n$ consists of n-tuples $(x_1, \ldots, x_n)$ of real numbers, with coordinatewise addition and multiplication by scalars. A set of elements $\{e_\alpha\}$ is a *basis* for a vector space if every element in the space has a unique expression in the form $\Sigma x_\alpha e_\alpha$, for some real numbers x_α with only finitely many x_α nonzero. A vector space is *finite dimensional* if it has a finite basis. The number of elements in a basis is independent of choice of basis, and is the *dimension* of the space; the dimension of V is denoted $\dim(V)$. Choosing a basis $e_1, \ldots, e_n$ for V sets up an isomorphism of V with $\mathbb{R}^n$, with the vector $x_1e_1 + \ldots + x_ne_n$ in V corresponding to $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$. The *standard basis* of $\mathbb{R}^n$ is the basis $\{e_i\}$ where e_i has a 1 for its ith coordinate, and zeros for the other coordinates.

If $L\colon V \to W$ is a linear mapping, the *kernel* $\mathrm{Ker}(L)$ is the subspace of V consisting of vectors mapped to zero, and the *image* $\mathrm{Im}(L)$ is the subspace of W consisting of vectors that can be written $L(v)$ for some v in V.

The *rank-nullity theorem* asserts that if $L\colon V \to W$ is a linear mapping of finite-dimensional vector spaces,

$$\dim(\mathrm{Ker}(L)) + \dim(\mathrm{Im}(L)) = \dim(V).$$

If W is a subspace of a vector space V, the *quotient space* V/W is defined to be the set of equivalence classes of elements of V, two vectors in V being equivalent if their difference is in W. This set V/W has a natural structure of a vector space, so that the mapping from V to V/W that takes a vector to its equivalence class is a linear mapping of vector spaces. The kernel of this mapping from V to V/W is W.

Conversely, if $V \to U$ is a surjective linear mapping of vector spaces, and W is the kernel, this determines an isomorphism of V/W with U.

Suppose $L: V \to V'$ is a linear mapping of vector spaces, and W is a subspace of V, and W' a subspace of V'. If $L(W)$ is contained in W', then L determines a linear mapping.

$$V/W \to V'/W'$$

of quotient spaces, which takes the class of v in V to the class of $L(v)$ in V'.

If V and W are vector spaces, the *direct sum* $V \oplus W$ can be defined as the set of pairs (v, w), with v in V and w in W, with addition defined by $(v, w) + (v', w') = (v + v', w + w')$, and multiplication by scalars by $r \cdot (v, w) = (r \cdot v, r \cdot w)$. For example, $\mathbb{R}^n$ is the direct sum of n copies of $\mathbb{R}$. More generally, given any collection V_α of vector spaces, for α in some index set $\mathscr{A}$, an element of direct sum $\oplus V_\alpha$ is determined by specifying a vector v_α in V_α for each α in $\mathscr{A}$, with the added condition that v_α can be nonzero for only finitely many α. Addition and multiplication by scalars are defined component by component, as for two factors. The same definition, but without the restriction that only finitely many are nonzero, defines the *direct product*, denoted ΠV_α.

For vectors $u = (x_1, \ldots, x_n)$ and $v = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, the *dot product* is the number $u \cdot v = x_1 y_1 + \ldots + x_n y_n$. The *length* of u is $\|u\| = \sqrt{u \cdot u}$. The *projection* of u on v is the vector tv, where $t = (u \cdot v)/(v \cdot v)$; the length of this projection is $|u \cdot v|/\|v\|$.

An m by n matrix $A = (a_{i,j})$ determines a linear mapping $L: \mathbb{R}^n \to \mathbb{R}^m$ that takes e_j to $L(e_j) = \Sigma_{i=1}^m a_{i,j} e_i$. Every linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ arises from a unique such matrix. If $M: \mathbb{R}^m \to \mathbb{R}^l$ corresponds to an l by m matrix $B = (a_{j,k})$, the composite $M \circ L: \mathbb{R}^n \to \mathbb{R}^l$ corresponds to the *product* matrix $B \cdot A$, where the (i, j) entry of $B \cdot A$ is $\Sigma_{k=1}^m b_{i,k} a_{k,j}$. We need this mainly for (2×2) matrices, where a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a matrix, the linear mapping corresponding to A takes a vector $v = (x, y)$ to the vector $(ax + by, cx + dy)$. The determinant of A, denoted $\det(A)$, is $ad - bc$. If the determinate is nonzero, A is invertible, with inverse

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The invertible (2×2)-matrices form a group, denoted $\mathrm{GL}_2 \mathbb{R}$. This group has a topology, determined by its embedding as an open subset of $\mathbb{R}^4$: the complement of the set of (a, b, c, d) with $ad - bc = 0$.

Exercise C.1. Show that the product $\mathrm{GL}_2 \mathbb{R} \times \mathrm{GL}_2 \mathbb{R} \to \mathrm{GL}_2 \mathbb{R}$, $A \times B \mapsto A \cdot B$, and the inverse map $\mathrm{GL}_2 \mathbb{R} \to \mathrm{GL}_2 \mathbb{R}$, $A \mapsto A^{-1}$, are continuous mappings.

Lemma C.2. *If* $det(A) > 0$, *there is a path* $\gamma\colon [a, b] \to GL_2\mathbb{R}$ *such that* $\gamma(a) = A$ *and* $\gamma(b) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. *If* $det(A) < 0$, *there is a path* $\gamma\colon [a, b] \to GL_2\mathbb{R}$ *such that* $\gamma(a) = A$ *and* $\gamma(b) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Proof. We will find a sequence of paths, each taking the matrix to a simpler one. For example, by multiplying a column of a matrix by t, with t varying in $[\alpha, \beta]$, for α and β positive, we can find a path changing the lengths of the column. In particular, we can assume the first column of A is a unit vector, so it can be written in the form $(\cos(\vartheta), \sin(\vartheta))$ for some ϑ in $[0, 2\pi]$. Then the path

$$\gamma(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \cdot A, \qquad 0 \leq t \leq \vartheta,$$

takes A to a matrix whose first column is $(1, 0)$. Then one can gradually project the second column on the line perpendicular to the first, via the path

$$\gamma(t) = \begin{bmatrix} 1 & (1-t)b \\ 0 & d \end{bmatrix}, \qquad 0 \leq t \leq 1,$$

to get to a matrix where the second column is $(0, d)$. Changing the length of the second column as at the beginning, we can get it either to $(0, 1)$ or to $(0, -1)$, as asserted. □

Problem C.3. Generalize to $GL_n\mathbb{R}$, showing that $GL_n\mathbb{R}$ has two connected components for all $n \geq 1$.

C2. Groups; Free Abelian Groups

A set of elements in a group G *generates* G if every element in G can be written as a (finite) product of elements in the set and inverses of elements in the set.

If H is a subgroup of a group G, a *left coset* is a subset of G of the form $g \cdot H = \{g \cdot h\colon h \in H\}$. The group G is a disjoint union of its left cosets; g_1 and g_2 are in the same left coset exactly when there is an element h in H with $g_1 \cdot h = g_2$. The set of left cosets is denoted by G/H. There is a natural map π from G onto G/H that takes an element in G to the coset containing it. A subgroup H is a *normal* subgroup if, for all g in G and h in H, $g \cdot h \cdot g^{-1}$ is in H. In this case G/H gets the structure of a group, in such a way that the natural map $\pi\colon G \to G/H$ is a homomorphism of groups.

The identity element in a group G is usually denoted by e, or e_G if there is chance of confusion. If $\varphi\colon G \to G'$ is a homomorphism of groups, the *kernel* $N = \{g \in G\colon \varphi(g) = e_{G'}\}$ is a normal subgroup of G, denoted $\mathrm{Ker}(\varphi)$. Then φ determines a one-to-one homomorphism $\overline{\varphi}\colon G/\mathrm{Ker}(\varphi) \to G'$ such that

$\varphi = \overline{\varphi} \circ \pi$. If φ is surjective, then $\overline{\varphi}$ is an isomorphism. More generally, if N is any normal subgroup of G, a homomorphism from G/N to a group G' determines a homomorphism from G to G' such that N is contained in its kernel. The *image* $\varphi(G)$ of any homomorphism $\varphi: G \to G'$ is a subgroup of G', denoted $\mathrm{Im}(\varphi)$, and φ determines an isomorphism of $G/\mathrm{Ker}(\varphi)$ with $\mathrm{Im}(\varphi)$. If $N \subset G$ and $N' \subset G'$ are normal subgroups, and $\varphi: G \to G'$ is a homomorphism such that $\varphi(N) \subset N'$, then φ determines a homomorphism $\overline{\varphi}: G/N \to G'/N'$.

The set of homomorphisms from G to G' is denoted by $\mathrm{Hom}(G, G')$. So if N is a normal subgroup of G,

$$\mathrm{Hom}(G/N, G') \leftrightarrow \{\varphi \in \mathrm{Hom}(G, G'): \varphi(N) = e_{G'}\}.$$

An important normal subgroup of a group G is the *commutator subgroup*, denoted $[G, G]$. This consists of all finite products

$$g_1 h_1 g_1^{-1} h_1^{-1} \cdot g_2 h_2 g_2^{-1} h_2^{-1} \cdot \ldots \cdot g_n h_n g_n^{-1} h_n^{-1},$$

for elements $g_1, h_1, g_2, h_2, \ldots, g_n, h_n$ in G. The normality of this subgroup comes from the identity $g \cdot (ab) \cdot g^{-1} = (g \cdot a \cdot g^{-1}) \cdot (g \cdot b \cdot g^{-1})$. If A is an abelian group, any homomorphism of G to A sends all commutators to the identity, so

$$\mathrm{Hom}(G/[G, G], A) \leftrightarrow \mathrm{Hom}(G, A).$$

We usually use an *additive* notation for the product in abelian groups, writing $g + h$ instead of $g \cdot h$, with the identity element denoted 0. The group of integers under addition, which is the infinite cyclic group, is denoted $\mathbb{Z}$. The abelian group with just one element is often denoted 0. If A is an abelian group, and X is any set, the set of functions from X to A has a natural structure of abelian group, with $(f + g)(x) = f(x) + g(x)$. In particular, if G is any group, the set of homomorphisms $\mathrm{Hom}(G, A)$ has the structure of an abelian group.

Exercise C.4. If $\varphi: G \to G'$ is a homomorphism of groups, show that the mapping from $\mathrm{Hom}(G', A)$ to $\mathrm{Hom}(G, A)$ that takes ψ to $\psi \circ \varphi$ is a homomorphism of abelian groups.

If A and B are abelian groups, the *direct sum* $A \oplus B$ is the set of pairs (a, b), with a in A and b in B, with addition defined by $(a, b) + (a', b') = (a + a', b + b')$. More generally, given any collection A_α of abelian groups, the *direct sum* $\oplus A_\alpha$ consists of collections $\{a_\alpha\}$, with a_α in A_α, with the condition that a_α can be nonzero for only finitely many α. Addition is defined component by component, as for two factors. For example, $\mathbb{Z}^n$ is the direct sum of n copies of $\mathbb{Z}$. The same definition, but without the restriction that only finitely many are nonzero, defines the *direct product*, denoted $\prod A_\alpha$.

Exercise C.5. If an abelian group C contains subgroups A and B such that every element of C can be written as a sum of an element in A and an element in B, and $A \cap B = \{0\}$, show that $A \oplus B$ is isomorphic to C.

Exercise C.6. For any collection A_α of abelian groups, and any abelian group B, construct an isomorphism

$$\operatorname{Hom}(\oplus A_\alpha, B) \cong \Pi \operatorname{Hom}(A_\alpha, B).$$

An abelian group A is a *free abelian group*, with *basis* $\{e_\alpha\}$, if every element in the group has a unique expression in the form $\Sigma n_\alpha e_\alpha$, for some integers n_α, with only finitely many n_α nonzero. If the number of elements in a basis is a finite number n, we say A is a free abelian group of *rank n*. As we will see below, this number is independent of choice of basis.

Exercise C.7. (a) If A and B are free abelian groups, show that $A \oplus B$ is free abelian, and if the ranks are finite, $\operatorname{rank}(A \oplus B) = \operatorname{rank}(A) + \operatorname{rank}(B)$. (b) If F is free abelian, and A is abelian, and $\varphi\colon A \to F$ is a surjective homomorphism, show that A is isomorphic to the direct sum of F and $\operatorname{Ker}(\varphi)$.

More generally, a set $\{e_\alpha\}$ of elements in an abelian group A is called *linearly independent* if no linear combination of them is zero, i.e., there is no set of integers $\{n_\alpha\}$, with only finitely many nonzero, but not all zero, such that $\Sigma n_\alpha e_\alpha = 0$. The maximum number of elements in a linearly independent set in A is called the *rank* of A. We will prove at the end of this section that any two maximal linearly independent sets have the same number of elements, at least when this number is finite.

Unlike the case with vector spaces, however, a maximal set of linearly independent elements in an abelian group need not generate the group. For example, a finite abelian group has no independent elements. Even for groups with no elements of finite order, however, it is not true:

Exercise C.8. Show that the rank of the abelian group $\mathbb{Q}$ of rational numbers is 1.

For any set X, the *free abelian group* on X, denoted $F(X)$, can be defined as the set of finite formal linear combinations $\Sigma n_x x$, with n_x integers, the sum over a finite subset of X. The addition is defined coordinate-wise: $\Sigma n_x x + \Sigma m_x x = \Sigma(n_x + m_x)x$. More precisely, define $F(X)$ to be the set of functions from X to $\mathbb{Z}$ that are zero except on a finite subset of X. This is an abelian subgroup of the abelian group of functions from X to $\mathbb{Z}$. To the function $f\colon X \to \mathbb{Z}$ is associated the expression $\Sigma f(x)x$. The element x corresponds to the function that takes x to 1 and all other elements of X to zero. These elements form a basis for $F(X)$.

For any abelian group A, and any function φ from a set X to A, there is a unique homomorphism from $F(X)$ to A that takes $\Sigma n_x x$ to $\Sigma n_x \varphi(x)$. In particular, if $\varphi\colon X \to Y$ is any function, it determines a homomorphism from

$F(X)$ to $F(Y)$, taking $\Sigma n_x x$ to $\Sigma n_x \varphi(x)$, or taking $n_1 x_1 + \ldots + n_r x_r$ to $n_1\varphi(x_1) + \ldots + n_r\varphi(x_r)$.

Exercise C.9. If $\varphi\colon X \to Y$ is one-to-one, show that $F(X) \to F(Y)$ is one-to-one, and if $\varphi\colon X \to Y$ is surjective, show that $F(X) \to F(Y)$ is surjective.

If A is an abelian group, then the set of homomorphisms $\operatorname{Hom}(A, \mathbb{R})$ from A to $\mathbb{R}$ forms a real vector space, with scalar multiplication by the rule $(rf)(x) = r \cdot f(x)$ for f in $\operatorname{Hom}(A, \mathbb{R})$, r a real number, and x in A. If $\varphi\colon A \to A'$ is a homomorphism of abelian groups, then there is a linear mapping $\varphi^*\colon \operatorname{Hom}(A', \mathbb{R}) \to \operatorname{Hom}(A, \mathbb{R})$ of vector spaces, defined by the formula $\varphi^*(f) = f \circ \varphi$.

Lemma C.10. *If $\varphi\colon A \to A'$ is one-to-one, then φ^* is surjective.*

Proof. We need some preliminaries. There exists a set $\mathcal{B} = \{x_\alpha\colon \alpha \in \mathcal{A}\}$ of elements in A', such that:

(i) no finite linear combination $\Sigma n_\alpha x_\alpha$ with integer coefficients is in the image of φ unless all n_α are zero; and
(ii) $\mathcal{B}$ is maximal with this property.

This is a consequence of Zorn's lemma, exactly as in the proof that every vector space has a basis. Of course, there may be many such sets $\mathcal{B}$, but we fix one. It follows that for any element x in A', there is a nonzero integer n and integers n_α, all zero except for finitely many, so that $nx - \Sigma n_\alpha x_\alpha$ is in $\varphi(A)$; otherwise one could enlarge $\mathcal{B}$ by adding x to it. Therefore for any x in A' there is at least one equation of the form

$$nx = \Sigma n_\alpha x_\alpha + \varphi(y) \tag{C.11}$$

with y in A, n not 0.

Given f in $\operatorname{Hom}(A, \mathbb{R})$, we define g in $\operatorname{Hom}(A', \mathbb{R})$ by setting

$$g(x) = \frac{1}{n}(f(y)),$$

for any integer $n \neq 0$ and $y \in A$ so that (C.11) holds. To see that g is well defined, suppose also

$$mx = \Sigma m_\alpha x_\alpha + \varphi(z),$$

with $z \in A$ and $m \neq 0$. Then

$$\begin{aligned} \Sigma (mn_\alpha - nm_\alpha)x_\alpha &= (mnx - m\varphi(y)) - (nmx - n\varphi(z)) \\ &= \varphi(nz) - \varphi(my) = \varphi(nz - my). \end{aligned}$$

By (i), this element must be zero, and since φ is one-to-one, nz must be equal to my. Therefore,

$$\frac{1}{m}(f(z)) = \frac{1}{mn}(f(nz)) = \frac{1}{mn}(f(my)) = \frac{1}{mn}(mf(y)) = \frac{1}{n}(f(y)),$$

as required.

The proof that g is a homomorphism is similar, for if x and x' are two elements of A', write

$$nx = \Sigma n_\alpha x_\alpha + \varphi(y) \quad \text{and} \quad mx' = \Sigma m_\alpha x_\alpha + \varphi(z).$$

Then $mn(x \pm x') = \Sigma(mn_\alpha \pm nm_\alpha)x_\alpha + \varphi(my \pm nz)$, so

$$g(x \pm x') = \frac{1}{mn}(f(my \pm nz)) = \frac{1}{n}f(y) \pm \frac{1}{m}f(z) = g(x) \pm g(x').$$

And, by definition, if $x = \varphi(y)$, then $g(x) = f(y)$, so $\varphi^*(g) = f$. □

Corollary C.12. *Suppose S is a set with n elements in an abelian group A, and S is a maximal set of linearly independent elements. Then the dimension of $Hom(A, \mathbb{R})$ is equal to n. In particular, any two maximal sets of linearly independent elements in A have the same number of elements.*

Proof. Let F be the free abelian group on S, and $\varphi: F \to A$ the natural map. The linear independence of S assures that φ is one-to-one. The maximality of S assures that the set $\mathcal{B}$ considered in the proof of the lemma is empty. The proof of the lemma shows that the map φ^* from $\mathrm{Hom}(A, \mathbb{R})$ to $\mathrm{Hom}(F, \mathbb{R})$ is an isomorphism. The functions that take value 1 on a given element in S, and value 0 on the other elements, give a basis of $\mathrm{Hom}(F, \mathbb{R})$ with n elements, and this shows that the dimension of $\mathrm{Hom}(A, \mathbb{R})$ is n. Note that if S were a maximal set of linearly independent elements that were infinite, the same argument shows that $\mathrm{Hom}(A, \mathbb{R}) \cong \mathrm{Hom}(F, \mathbb{R})$ is infinite dimensional. □

Exercise C.13. Show conversely that if $\mathrm{Hom}(A, \mathbb{R})$ has finite dimension n, then A has rank n.

Exercise C.14. If $B \to C$ is surjective, with kernel A, show that if two of the three abelian groups A, B, and C have finite ranks, so does the third, and $\mathrm{rank}(B) = \mathrm{rank}(A) + \mathrm{rank}(C)$.

Given homomorphism $\varphi: A \to B$ and $\psi: B \to C$, one says that the sequence $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ is *exact*, or *exact at B* if the image of φ is equal to the kernel of ψ. To say that the sequence $0 \to A \to B$ is exact is the same as saying the map from A to B is one-to-one, and to say that $A \to B \to 0$ is exact is the same as saying the map from A to B is surjective.

Problem C.15. If $A \to B \to C$ is exact at B, show that the dual sequence $\mathrm{Hom}(C, \mathbb{R}) \to \mathrm{Hom}(B, \mathbb{R}) \to \mathrm{Hom}(A, \mathbb{R})$ is exact at $\mathrm{Hom}(B, \mathbb{R})$.

A sequence $A_0 \to A_1 \to \ldots \to A_n \to A_{n+1}$ of abelian groups and homomorphisms between them is called *exact* if it is exact at each of the groups A_i, for $1 \le i \le n$.

Problem C.16. Show that if $0 \to A_1 \to \ldots \to A_n \to 0$ is exact, and each of the abelian groups A_i has finite rank, then

$$\sum_{i=1}^{n} (-1)^i \operatorname{rank}(A_i) = 0.$$

C3. Polynomials; Gauss's Lemma

For any field K the ring of polynomials $K[X]$ in a variable X over K is a *unique factorization domain.* In fact, every nonzero P in $K[X]$ has a unique factorization $P = a \cdot \Pi P_i^{n_i}$, with a in K and each P_i an irreducible polynomial that is *monic.*[9] This follows from the fact that one has a division algorithm for polynomials, just as one has for integers. In particular, any finite collection of nonzero polynomials has a greatest common divisor, which is unique if, in addition, it is required to be monic. The quotient field of $K[X]$, consisting of all ratios P/Q, $Q \neq 0$, is denoted $K(X)$.

The ring of polynomials $K[X, Y]$ in two variables X and Y is a subring of the ring $K(X)[Y]$, which, by what we have just seen, is a unique factorization domain.

Lemma C.17 (Gauss). *Let F be a polynomial in $K[X, Y]$. If F is irreducible in $K(X)[Y]$, then F is irreducible in $K[X, Y]$.*

Proof. Given F in $K[X, Y]$, write $F = a_0(X) + a_1(X)Y + \ldots + a_n(X)Y^n$, with $a_i(X)$ in $K[X]$. The greatest common divisor of $a_0(X), \ldots, a_n(X)$ is called the *content* of F, and denoted $c(F)$. Call F *primitive* if $c(F) = 1$.

We show first that the product of two primitive polynomials is also primitive. To see this, suppose $F = a_0 + a_1 Y + \ldots + a_n Y^n$ and $G = b_0 + b_1 Y + \ldots + b_m Y^m$ are primitive. Suppose a nonconstant polynomial $p = p(X)$ divides all the coefficients of $F \cdot G$. Take the minimal i and j such that p does not divide a_i and b_j. Then the coefficient of Y^{i+j} in $F \cdot G$ has the form

$$a_i b_j + a_{i+1} b_{j-1} + \ldots + a_{i+j} b_0 + a_{i-1} b_{j+1} + \ldots + a_0 b_{i+j}.$$

[9]A monic polynomial is one of the form $X^m + a_1 X^{m-1} + \ldots + a_m$.

Since all the terms but the first are divisible by p, and the first is not, this is a contradiction.

It follows from this that for any two polynomials F and G in $K[X, Y]$, $c(F \cdot G) = c(F) \cdot c(G)$. To see this, write $F = c(F) \cdot F_1$, $G = c(G) \cdot G_1$, with F_1 and G_1 primitive. Then $F \cdot G = c(F) \cdot c(G) \cdot F_1 \cdot G_1$, and $F_1 \cdot G_1$ is primitive, from which it follows that the content of $F \cdot G$ is $c(F) \cdot c(G)$.

Given any nonzero G in $K(X)[Y]$, one can write $G = g \cdot G_1$, with g in $K(X)$ and G_1 a primitive polynomial in $K[X, Y]$. Now suppose F is an irreducible polynomial in $K[X, Y]$, and that F factors in $K(X)[Y]$ into $G \cdot H$, with both G and H of positive degree in Y. Write $G = g \cdot G_1$, $H = h \cdot H_1$, with G_1 and H_1 primitive, and $g = p/q$, $h = r/s$, with $p, q, r, s \in K[X]$. Then $q \cdot s \cdot F = p \cdot r \cdot G_1 \cdot H_1$ in $K[X, Y]$. It follows that $q \cdot s \cdot c(F) = p \cdot r$. Hence $F = c(F) \cdot G_1 \cdot H_1$, which contradicts the irreducibility of F in $K[X, Y]$. $\square$

Exercise C.18. Show that, for any field K, $K[X, Y]$ is a unique factorization domain. Generalize to polynomials in n variables.

If P is a polynomial in $K[X]$, the residue class ring $K[X]/(P)$ is the set of equivalence classes of polynomials in $K[X]$, two being equivalent when their difference is divisible by P. The residue classes $K[X]/(P)$ have the structure of a ring so that the natural map $K[X] \to K[X]/(P)$ is a homomorphism of rings.

Lemma C.19. *If P has degree n, then the images of $1, X, \ldots, X^{n-1}$ form a basis for $K[X]/(P)$ over K.*

Proof. These elements span, since, by dividing by P, any polynomial is equivalent to a polynomial of degree less than n. They are linearly independent, since no nonzero polynomial of degree less than n is divisible by P. $\square$

Exercise C.20. Show that $K[X]/(P)$ is a field if and only if P is irreducible.

APPENDIX D

On Surfaces

D1. Vector Fields on Plane Domains

The object of this section is to prove Lemmas 7.10 and 7.11; we refer to Chapter 7 for notation. Suppose $\varphi\colon U \to U'$ is a diffeomorphism from one open set in the plane onto another. If $\varphi(x,y) = (u(x,y), v(x,y))$ in coordinates, at any point P in U, we have the Jacobian matrix

$$J_{\varphi,P} = \begin{bmatrix} \dfrac{\partial u}{\partial x}(P) & \dfrac{\partial u}{\partial y}(P) \\ \dfrac{\partial v}{\partial x}(P) & \dfrac{\partial v}{\partial y}(P) \end{bmatrix},$$

which we regard as a linear mapping from vectors in $\mathbb{R}^2$ to vectors in $\mathbb{R}^2$ (see Appendix C). If V is a continuous vector field in U, the vector field φ_*V in U' is defined by the formula

$$(\varphi_*V)(P') = J_{\varphi,P}(V(P)),$$

where P is the point in U mapped to P' by φ, i.e., $P = \varphi^{-1}(P')$. If V has singularities in the set Z, φ_*V will have singularities in $\varphi(Z)$.

Lemma D.1. $\mathrm{Index}_{\varphi(P)}(\varphi_*V) = \mathrm{Index}_P V$.

Proof. There is no loss in generality by assuming that P and P' are the origin 0, that U is a disk containing the origin, and that V is not zero in $U \setminus \{0\}$.

Let J be the Jacobian of φ at 0. Our first goal is to show that

$$\text{(D.2)} \qquad \operatorname{Index}_0(\varphi_* V) = \operatorname{Index}_0(J_* V).$$

This will reduce the problem to the easier case of a linear mapping. We want to construct a homotopy from φ to J. Define

$$K\colon U \times [0,1] \to \mathbb{R}^2, \qquad Q \times t \mapsto \begin{cases} \dfrac{1}{t}\varphi(t \cdot Q), & 0 < t \leq 1, \\ J(Q), & t = 0. \end{cases}$$

Claim D.3. *This mapping K is $\mathscr{C}^\infty$.*

To prove this claim, we use Lemma B.9, replacing U by a smaller disk if necessary, so we can write

$$\varphi(x,y) = (xu_1(x,y) + yu_2(x,y), xv_1(x,y) + yv_2(x,y)),$$

with $\mathscr{C}^\infty$ functions u_1, u_2, v_1, and v_2. Then

$$K((x,y)\times t) = (xu_1(tx,ty) + yu_2(tx,ty), xv_1(tx,ty) + yv_2(tx,ty))$$

for all $0 \leq t \leq 1$, and this expression is clearly $\mathscr{C}^\infty$.

Now $H(Q \times t) = (K_t)_*(V)$ gives a homotopy from J_*V to φ_*V in the sense of Exercise 7.3, and (D.2) follows from that exercise.

We are therefore reduced to showing that $\operatorname{Index}_0(J_*V) = \operatorname{Index}_0(V)$ for any invertible linear mapping J. Now we use Lemma C.2 to know that there is a path in the space of invertible matrices from J either to the identity matrix I, or to the matrix $I' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. If such a path is given by a formula $t \mapsto J_t$, $a \leq t \leq b$, then the homotopy $H(Q \times t) = (J_t)_*(V)$ gives a homotopy from J_*V to I_*V or to I'_*V, and the same exercise shows that the index doesn't change. Of course $I_*V = V$, so all that remains is to prove that $\operatorname{Index}_0(I'_*V) = \operatorname{Index}_0(V)$.

If $V(x,y) = (p(x,y), q(x,y))$, then by the definition of I'_*,

$$(I'_*V)(x,y) = (p(x,-y), -q(x,-y)).$$

So one is reduced to the elementary problem of showing that if $F(x,y) = (p(x,y), q(x,y))$, and $R(x,y) = (x,-y)$, the mappings $R \circ F \circ R$ and F, when restricted to a small circle, have the same winding number around the origin. This is easy to do directly from the definition of winding number, and we leave the details as an exercise. (This is also special case of the fact that the degree of a composite of mappings of circles is the product of the degrees of the mappings, see Problem 3.27. In this way one can argue directly with any linear mapping J, since $J_*V = J \circ V \circ J^{-1}$.) $\square$

Now we consider the other lemma from Chapter 7.

Lemma D.4. *Suppose V and W are continuous vector fields with no singularities on an open set U containing a point P. Let $D \subset U$ be a closed disk centered at P. Then there is a vector field $\tilde{V}$ with no singularities on U such that* (*i*) *$\tilde{V}$ and V agree on $U \setminus D$;* (*ii*) *$\tilde{V}$ and W agree on some neighborhood of P.*

Proof. Suppose first that the dot product $V(P) \cdot W(P)$ is positive. Shrinking the disk, we may assume that $V(Q) \cdot W(Q) > 0$ for all Q in D. As in Step 2 of the proof of Proposition B.10 (see Exercise B.14), there is a $\mathscr{C}^\infty$ function ρ that is identically 1 in a neighborhood of P, and identically 0 outside D, and taking values in $[0, 1]$. Let

$$\tilde{V}(Q) \;=\; (1 - \rho(Q))V(Q) + \rho(Q)W(Q).$$

Then $V(Q) \cdot \tilde{V}(Q) > 0$ for all Q in U, so $\tilde{V}$ has no singularities, and conditions (i) and (ii) are clear.

For the general case, it therefore suffices to find a vector field V_1 with no singularities that agrees with V outside D, and such that $V_1(P) \cdot W(P)$ is positive. This can be done by rotating V inside D. With the same function ρ, and ϑ the angle from the vector $V(P)$ to the vector $W(P)$, we can take

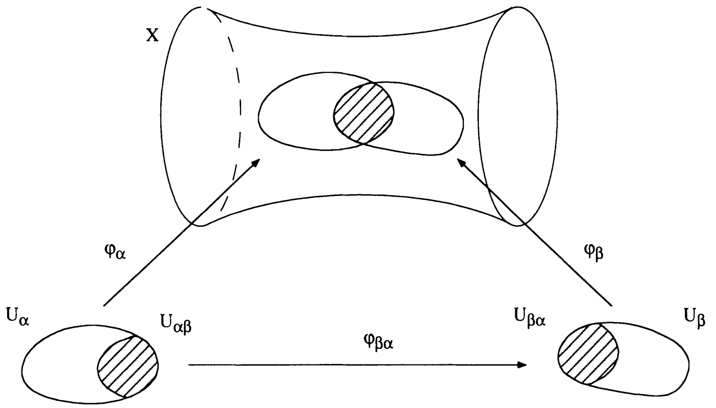
$$V_1(Q) \;=\; \begin{bmatrix} \cos(\rho(Q)\vartheta) & -\sin(\rho(Q)\vartheta) \\ \sin(\rho(Q)\vartheta) & \cos(\rho(Q)\vartheta) \end{bmatrix} \cdot V(Q). \qquad \square$$

D2. Charts and Vector Fields

We start with a brief definition of a (smooth) surface X, and define what we mean by a vector field on X and the index of a vector field at a point on X. A *surface X with an atlas of charts* is a Hausdorff topological space, equipped with a collection of homeomorphisms

$$\varphi_\alpha \colon U_\alpha \to \varphi_\alpha(U_\alpha) \subset X,$$

with U_α open in the plane $\mathbb{R}^2$, and $\varphi_\alpha(U_\alpha)$ open in X; the α are in some index set. The surface X should be the union of these open sets $\varphi_\alpha(U_\alpha)$. Let $U_{\alpha\beta} = {\varphi_\alpha}^{-1}(\varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta))$. These charts determine *change of coordinate mappings* $\varphi_{\beta\alpha} = {\varphi_\beta}^{-1} \circ \varphi_\alpha$, which are homeomorphisms from $U_{\alpha\beta}$ to $U_{\beta\alpha}$.



To give X a differentiable, or smooth, structure, the requirement is that these changes of coordinates $\varphi_{\beta\alpha}$ should be $\mathscr{C}^\infty$ mappings for all α and β. One then has a notion of a *differentiable function* on an open subset U of X: it is function f such that $f \circ \varphi_\alpha$ is differentiable on $\varphi_\alpha{}^{-1}(U) \cap U_\alpha$ for all α.

Another collection of charts $\{\psi_{\alpha'}: U_{\alpha'} \to X\}$ is said to be equivalent to this one if all the changes of coordinates from one to the other are all $\mathscr{C}^\infty$, i.e., all $\varphi_\alpha{}^{-1} \circ \psi_{\alpha'}$ are $\mathscr{C}^\infty$ where defined. We say that this collection defines the same surface. More precisely, a *smooth* (or $\mathscr{C}^\infty$) *surface* is the topological space X together with an equivalence class of families of charts. For the sphere S^2, the two mappings φ and ψ we obtained from stereographic projection in §7c form a family of charts. Stereographic projection from other points gives charts that are compatible in this sense.

If f is a $\mathscr{C}^\infty$ function on some open set in $\mathbb{R}^3$, and X is the locus where $f(x,y,z) = 0$ and $\operatorname{grad}(f) \neq 0$, then X is a smooth surface. If for example $(\partial f/\partial z)(P) \neq 0$, the implicit function theorem says that projection from X to the xy-plane is locally one-to-one near P, and the inverse to this projection provides a chart near P.

If X is a surface given by charts as above, a *vector field* V on X can be defined as a compatible collection of vector fields V_α in each of the coordinate neighborhoods U_α. The compatibility means that

$$(\varphi_{\beta\alpha})_*(V_\alpha|_{U_{\alpha\beta}}) = V_\beta|_{U_{\beta\alpha}},$$

where $(\varphi_{\beta\alpha})_*$ is defined as in the preceding section. The vector field has a singularity at a point P in X if, with some α such that $\varphi_\alpha(P_\alpha) = P$, the corresponding V_α has a singularity at P_α in U_α. By Lemma D.1 the index of V_α at P_α is independent of choice of coordinate chart. This index is defined to be the *index* of V at P.

The surface X is *orientable* if it has an atlas of charts such that all the determinants of the Jacobians of the change of coordinate mappings are positive. An orientation is a choice of such an atlas, with two atlases defining the same orientation if all changes of coordinates from one to the other have positive determinants of Jacobians. An orientable surface has two orientations, with two orientations defining the opposite orientation if all changes of coordinates from one to the other have negative Jacobian determinants.

D3. Differential Forms on a Surface

If $\varphi\colon U \to U'$ is a diffeomorphism from one open set in the plane to another, and $\omega = f\,dx + g\,dy$ is a $\mathscr{C}^\infty$ 1-form on U', one can define a *pull-back* 1-form $\varphi^*\omega$ on U by the formula

$$\begin{aligned}\varphi^*(f\,dx + g\,dy) &= f(\varphi(x,y))\cdot\left(\frac{\partial u}{\partial x}(x,y)\,dx + \frac{\partial u}{\partial y}(x,y)\,dy\right)\\ &\qquad + g(\varphi(x,y))\cdot\left(\frac{\partial v}{\partial x}(x,y)\,dx + \frac{\partial v}{\partial y}(x,y)\,dy\right)\\ &= \left((f\circ\varphi)\cdot\frac{\partial u}{\partial x} + (g\circ\varphi)\cdot\frac{\partial v}{\partial x}\right)dx\\ &\qquad + \left((f\circ\varphi)\cdot\frac{\partial u}{\partial y} + (g\circ\varphi)\cdot\frac{\partial v}{\partial y}\right)dy,\end{aligned}$$

where $\varphi(x,y) = (u(x,y), v(x,y))$. Similarly, if $\omega = h\,dx\,dy$ is a 2-form on U', the *pull-back* 2-form $\varphi^*\omega$ on U is defined by the formula

$$\varphi^*(h\,dx\,dy) = (h\circ\varphi)\cdot\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)dx\,dy.$$

Exercise D.5. (a) If $\psi\colon U' \to U''$ is a diffeomorphism, and ω is a 1-form or 2-form on U'', show that $\varphi^*(\psi^*\omega) = (\psi\circ\varphi)^*(\omega)$. (b) Show that $\varphi^*(df) = d(\varphi^*f)$ for a $\mathscr{C}^\infty$ function f on U', and $\varphi^*(d\omega) = d(\varphi^*\omega)$ for ω a 1-form on U'.

Given a surface X with an atlas of charts $\varphi_\alpha\colon U_\alpha \to X$ as in §D2, a function (or 0-form) is given by a collection of functions f_α on U_α such that they agree on the overlaps: $f_\alpha = f_\beta\circ\varphi_{\beta\alpha}$ on $U_{\alpha\beta}$. Define a *one-form* ω on X to be a collection of 1-forms ω_α on U_α that agree on the overlaps, i.e., such that

$$\omega_\alpha|_{U_{\alpha\beta}} = (\varphi_{\beta\alpha})^*(\omega_\beta|_{U_{\beta\alpha}})$$

for all pairs α and β. A *two-form* is defined likewise, taking the ω_α to be 2-forms on U_α.

If f is a $\mathscr{C}^\infty$ function (or *zero-form*) on X, its *differential* df is a 1-form, defined to be the 1-form $d(f\circ\varphi_\alpha)$ on U_α. Similarly, if ω is a 1-form on X, given by 1-forms ω_α on U_α, the *differential* of ω is the 2-form $d\omega$ defined to be the 2-form $d\omega_\alpha$ on U_α.

Exercise D.6. (a) Verify that these formulas define 1-forms and 2-forms on X. (b) Verify that d is linear, i.e., $d(r_1\omega_1 + r_2\omega_2) = r_1 d(\omega_1) + r_2 d(\omega_2)$ for real numbers r_1 and r_2 and 0-forms or 1-forms ω_1 and ω_2. (c) Verify that $d(df) = 0$. (d) Show that for $k = 0$, 1, and 2, a k-form for one atlas determines a k-form for any other atlas, and that this is compatible with the definition of differential.

There is also a *wedge product* $\wedge$ that takes two 1-forms ω and μ and produces a 2-form $\omega \wedge \mu$. For an open set U in the plane, if $\omega = f\,dx + g\,dy$, and $\mu = h\,dx + k\,dy$, for f, g, h, and k $\mathscr{C}^\infty$ functions on U, then

$$\omega \wedge \mu \;=\; (f\,dx + g\,dy) \wedge (h\,dx + k\,dy) = (f \cdot k - g \cdot h)\,dx\,dy\,.$$

If ω and μ are 1-forms on X given on U_α by ω_α and μ_α, respectively, then $\omega \wedge \mu$ is the 2-form given on U_α by $\omega_\alpha \wedge \mu_\alpha$, with $\omega_\alpha \wedge \mu_\alpha$ defined by the displayed formula. If f is a function and ω is a 1-form (or 2-form), then $f \cdot \omega$ (defined locally by $f_\alpha \cdot \omega_\alpha$) is a 1-form (or 2-form).

Exercise D.7. (a) Verify that $\omega \wedge \mu$ is a 2-form. (b) Verify the following properties of the wedge product:

(i) $$(f_1\omega_1 + f_2\omega_2) \wedge \mu \;=\; f_1(\omega_1 \wedge \mu) + f_2(\omega_2 \wedge \mu)$$

for ω_1, ω_2, and μ 1-forms, and f_1 and f_2 functions;

(ii) $$\mu \wedge \omega \;=\; -\omega \wedge \mu$$

for μ and ω 1-forms; and

(iii) $$d(f \cdot \mu) \;=\; (df) \wedge \mu + f \cdot d\mu$$

for f a function and μ a 1-form.

In fact, all the results of this appendix generalize from two to n dimensions, leading to the notion of a smooth manifold of dimension n, a vector field on a manifold, the index of a vector field, an orientation of a manifold, k-forms on a manifold ($0 \le k \le n$), with differential from k-forms to $(k + 1)$-forms, wedge products of k-forms and l-forms being $(k + l)$-forms, with similar properties.

APPENDIX E

Proof of Borsuk's Theorem

This appendix contains a proof of Borsuk's theorem as stated in §23c. It assumes a knowledge of §23a and §23b.

So far in this book we have considered chains $\Sigma n_i \Gamma_i$ with integer coefficients n_i. In fact, one can use coefficients in any abelian group G, and one gets chains $C_k(X; G)$, cycles $Z_k(X; G)$, and boundaries $B_k(X; G)$, so homology groups $H_k(X; G) = Z_k(X; G)/B_k(X; G)$. All the formal properties proved about ordinary homology groups extend without change to these groups, and many of the calculations are similar. For example, the Mayer–Vietoris theorem is true without change.

It is often useful to look at coefficients in $\mathbb{Z}/p\mathbb{Z}$, the integers modulo a prime p. In this case there is a natural homomorphism from each $H_k(X)$ to $H_k(X; \mathbb{Z}/p\mathbb{Z},)$ obtained by reducing all coefficients modulo p. This gives homomorphisms

$$H_k(X)/pH_k(X) \rightarrow H_k(X; \mathbb{Z}/p\mathbb{Z}).$$

These homomorphisms are isomorphisms if X is a sphere, as one sees by tracing through the Mayer–Vietoris argument computing the homology of a sphere. The following exercise shows that it is not always an isomorphism, however.

Exercise E.1. (a) Show that the above map is an isomorphism if X is a compact oriented surface. (b) Show that, for X the real projective plane, or any compact surface, $H_2(X; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

These homology groups are particularly useful when $p = 2$. In fact, every-thing becomes a little easier, since one can ignore all signs entirely. Since these are the only ones we will consider here, we denote them by $\overline{H}_kX$:

$$\overline{H}_kX = H_k(X;\mathbb{Z}/2\mathbb{Z}).$$

Similarly, we denote by $\overline{C}_*X$ the chain complex $C_*(X;\mathbb{Z}/2\mathbb{Z})$.

It follows from the isomorphism $H_n(S^n)/2H_n(S^n) \cong \overline{H}_n(S^n)$ that for any continuous map $f\colon S^n \to S^n$, the degree of f is odd if the homomorphism $f_*\colon \overline{H}_n(S^n) \to \overline{H}_n(S^n)$ is not zero, and even if f_* is zero. We denote the antipodal map by

$$T\colon S^n \to S^n.$$

Let $\pi\colon S^n \to \mathbb{RP}^n$ be the two-sheeted covering map that identifies antipodal points. We will prove Borsuk's theorem by using the chain complex and homology of these spaces and maps with coefficients in $\mathbb{Z}/2\mathbb{Z}$. For this we need a general lemma.

Let $\pi\colon X \to Y$ be any two-sheeted covering map, and let $T\colon X \to X$ be the map which interchanges the two points in $\pi^{-1}(P)$ for each P in Y. The corresponding map $\pi_*\colon \overline{C}_*X \to \overline{C}_*Y$ of chain complexes is surjective. In fact, if $\Gamma\colon I^k \to Y$ is a k-cube, there are exactly two k-cubes Λ_1 and $\Lambda_2 = T\circ\Lambda_1$ in X with $\pi\circ\Lambda_i = \Gamma$. This is proved exactly as we proved the lifting of homotopies in §11b. The mapping $\Gamma \mapsto \Lambda_1 + \Lambda_2$ determines a homomorphism $t_*\colon \overline{C}_*Y \to \overline{C}_*X$ of chain complexes, called the *transfer* (see Problem 18.26). A chain is in the kernel of π_* exactly when each cube $T\circ\Lambda$ occurs with the same coefficient as Λ. We therefore have an exact sequence of chain complexes

$$0 \to \overline{C}_*Y \xrightarrow{t_*} \overline{C}_*X \xrightarrow{\pi_*} \overline{C}_*Y \to 0. \tag{E.2}$$

This gives a long exact sequence in homology:

$$\ldots \to \overline{H}_kY \xrightarrow{t_*} \overline{H}_kX \xrightarrow{\pi_*} \overline{H}_kY \xrightarrow{\partial} \overline{H}_{k-1}Y \xrightarrow{t_*} \overline{H}_{k-1}X \to \ldots .$$

Lemma E.3. (*i*) *Let $f\colon X \to X$ be a continuous map such that $f\circ T = T\circ f$, and let $g\colon Y \to Y$ be the map determined by the condition $g\circ\pi = \pi\circ f$. Then the diagram*

$$\begin{tikzcd}
0 \arrow[r] & \overline{C}_*Y \arrow[r, "t_*"] \arrow[d, "g_*"] & \overline{C}_*X \arrow[r, "\pi_*"] \arrow[d, "f_*"] & \overline{C}_*Y \arrow[r] \arrow[d, "g_*"] & 0 \\
0 \arrow[r] & \overline{C}_*Y \arrow[r, "t_*"] & \overline{C}_*X \arrow[r, "\pi_*"] & \overline{C}_*Y \arrow[r] & 0
\end{tikzcd}$$

commutes.

(*ii*) *Let $f\colon X \to X$ be a continuous map such that $f\circ T = f$, and let $g\colon Y \to Y$ be the map determined by the condition $g\circ\pi = \pi\circ f$. Then the diagram*

$$\begin{tikzcd} 0 \arrow[r] & \bar{C}_*Y \arrow[r, "t_*"] \arrow[d, "0"'] & \bar{C}_*X \arrow[r, "\pi_*"] \arrow[d, "f_*"'] & \bar{C}_*Y \arrow[r] \arrow[d, "g_*"'] & 0 \\ 0 \arrow[r] & \bar{C}_*Y \arrow[r, "t_*"] & \bar{C}_*X \arrow[r, "\pi_*"] & \bar{C}_*Y \arrow[r] & 0 \end{tikzcd}$$

commutes.

Proof. Both of these are straightforward from the definitions. The right squares commute by functoriality: $g_* \circ \pi_* = (g \circ \pi)_* = (f \circ \pi)_* = f_* \circ \pi_*$. For the left squares, let Γ be a cube in Y, Λ_1 and Λ_2 its two liftings, so $f_*t_*[\Gamma] = [f \circ \Lambda_1] + [f \circ \Lambda_2]$. In case (i), $f \circ \Lambda_1$ and $f \circ \Lambda_2$ are the two liftings of $g \circ \Gamma$, so $t_*g_*[\Gamma] = [f \circ \Lambda_1] + [f \circ \Lambda_2]$, as required. In case (ii), $f \circ \Lambda_2 = f \circ T \circ \Lambda_1 = f \circ \Lambda_1$, so $f_*t_*[\Gamma] = 2[f \circ \Lambda_1] = 0$. □

It follows from this lemma that the corresponding maps between the long exact homology sequences commute. We apply this now to $X = S^n$, $Y = \mathbb{RP}^n$, $\pi\colon X \to Y$ the covering, and T the antipodal map. The long exact sequence arising from (E.2) takes the form

$$0 \longrightarrow \bar{H}_nY \xrightarrow{t_*} \bar{H}_nX \xrightarrow{\pi_*} \bar{H}_nY \xrightarrow{\partial} \bar{H}_{n-1}Y \longrightarrow 0$$

$$0 \longrightarrow \bar{H}_{n-1}Y \xrightarrow{\partial} \bar{H}_{n-2}Y \longrightarrow 0 \longrightarrow \dots$$

$$\dots \longrightarrow 0 \longrightarrow \bar{H}_1Y \xrightarrow{\partial} \bar{H}_0Y \xrightarrow{t_*} \bar{H}_0X \xrightarrow{\pi_*} \bar{H}_0Y \longrightarrow 0.$$

Since $\bar{H}_nX = \mathbb{Z}/2\mathbb{Z}$, we must have $\bar{H}_nY \neq 0$, so $t_*\colon \bar{H}_nY \to \bar{H}_nX$ is an isomorphism. Hence $\pi_*\colon \bar{H}_nX \to \bar{H}_nY$ is zero, so $\partial\colon \bar{H}_nY \to \bar{H}_{n-1}Y$ is an isomorphism. Continuing, we see that $\partial\colon \bar{H}_iY \to \bar{H}_{i-1}Y$ is an isomorphism for all $i = 1, \dots, n$. In particular, $\bar{H}_iY = \mathbb{Z}/2\mathbb{Z}$ for $i = 0, \dots, n$.

Now suppose $f\colon S^n \to S^n$ is a map with $f \circ T = T \circ f$. Borsuk's theorem (Theorem 23.24(a)) states that the degree of f is odd, which is equivalent to saying that $f_*\colon \bar{H}_nX \to \bar{H}_nX$ is an isomorphism. Applying Lemma E.3(i), we have commuting diagrams

$$\begin{tikzcd} \bar{H}_nY \arrow[r, "t_*", "\cong"'] \arrow[d, "g_*"'] & \bar{H}_nX \arrow[d, "f_*"'] \\ \bar{H}_nY \arrow[r, "t_*", "\cong"'] & \bar{H}_nX, \end{tikzcd} \qquad \begin{tikzcd} \bar{H}_iY \arrow[r, "\partial", "\cong"'] \arrow[d, "g_*"'] & \bar{H}_{i-1}Y \arrow[d, "g_*"'] \\ \bar{H}_iY \arrow[r, "\partial", "\cong"'] & \bar{H}_{i-1}Y, \end{tikzcd}$$

the right diagrams valid for $i = 1, \dots, n$. Since $g_*\colon \bar{H}_0Y \to \bar{H}_0Y$ is an isomorphism, it follows from the right squares and induction on i that the homomorphism $g_*\colon \bar{H}_iY \to \bar{H}_iY$ is an isomorphism for every $i = 0, \dots, n$. Then the left square implies that $f_*\colon \bar{H}_nX \to \bar{H}_nX$ is an isomorphism as well, and this completes the proof.

Similarly, if $f\colon S^n \to S^n$ is a map with $f \circ T = f$, we have by (ii) of the lemma a commutative diagram

$$\begin{array}{ccc} \bar{H}_n Y & \xrightarrow[\cong]{t_*} & \bar{H}_n X \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle f_*} \\ \bar{H}_n Y & \xrightarrow[\cong]{t_*} & \bar{H}_n X . \end{array}$$

This implies that $f_*: \bar{H}_n X \to \bar{H}_n X$ is zero, which means that f has even degree, and completes the proof of Theorem 23.24. □

The constructions of this section are part of a general development of P.A. Smith to study spaces equipped with periodic transformations like T. For a proof of Borsuk's theorem using simplicial approximations, see Armstrong (1983). For a proof using differential topology, see Guillemin and Pollack (1974).

Hints and Answers

Hints and/or answers are given for some of the exercises and problems, especially those used in the text, or those that are hard.

0.1. *Hint*: The answer depends only on the numbers of edges that emanate from each vertex. What happens to these numbers when you travel, erasing the edges as you travel over them? When do you get stuck at a vertex?

Answer: There is always an even number of vertices such that the number of edges emanating from the vertex is odd (if an edge has both ends at a vertex, it counts twice). If this number is greater than 2, the graph cannot be traced. If the number is 2, it can be traced, but only by starting at one of these, and (necessarily) ending at the other. If the number is 0, you can start anywhere, and will end at where you start. To see that one can do it under these conditions, one way is to make any trip, starting at an odd vertex if there are two such, continuing until you get stuck. Then make another trip, but adding a side trip along untraveled roads, until you (necessarily) get back to the old route at the same point. Each trip becomes longer, until the whole is traced.

0.2. See Chapter 8.

1.6. All but (vi).

1.7. Yes. Find such a function by integrating.

1.9. For the challenge, if P is in the closure of the points one can connect to P_0 by such an arc, take a disk D around P in U, take an are from P_0 to a point inside D, and look at the first time the arc hits the boundary of D with inward pointing tangent; splice on to this arc (see §B2 for similar constructions) to get to any point inside D.

1.13. See Chapter 9 for more general results.

1.20. For the challenge, use the law of cosines.

2.3. Show that the derivative of $\vartheta(t)$ must be

$$\frac{-y(t)x'(t)+x(t)y'(t)}{x(t)^2+y(t)^2}.$$

Or show that two such functions differ by a multiple of 2π, which must be 0 by continuity.

2.9. Consider neighborhoods given in polar coordinates by $\vartheta_1 < \vartheta < \vartheta_2$ and $r_1 < r < r_2$, with $\vartheta_2 - \vartheta_1 < 2\pi$.

2.13. See §B2 for the construction of such functions.

2.19. For formulas see Chapter 12, and use Problem 2.13.

2.22. Either argue directly, as in Appendix B, or use polar coordinates to map a rectangle onto the disk, and integrate the pull-back of the 1-form as in the first proof of Proposition 2.16.

2.24. Apply Green's formula (i) with $g = f$.

2.25. *Hint*: Apply Green's formula (ii), where R is the region inside the disk and outside a small disk around the point, with g of the form $a + b\log(r)$, where r is the distance from the center of the disk. Pass to the limit as the radius of the small disk approaches 0.

3.4. Use the definition. Choices of subdivision and sector U_i and ϑ_i for γ and P determine the same subdivision for $\gamma + v$, choices $U_i + v$ for sectors, and translated angle functions, so that the changes in angle along each piece are the same for each.

3.5. Use Exercise 2.9. See §11b for a generalization.

3.7. See Chapter 12 for formulas.

3.10. Apply the Lebesgue lemma to $\gamma \circ \varphi$, to obtain a subdivision $a' \le t_0' \le \ldots \le t_n' = b'$, such that $\gamma \circ \varphi$ maps each subinterval into a sector.

3.13. In the starshaped case, show that any closed path is homotopic to a constant path, and use Exercise 3.7.

3.15. Use Problem 3.14 to construct a homotopy between the lifted paths.

3.22. Use Problem 3.21 and Problem 3.14.

3.23. Use a homotopy $H(P \times s) = (1-s)F(P) \pm sP$, which is a homotopy from F to the mapping $P \mapsto P$ or to $P \mapsto -P$.

3.25. Part (d) uses Problem 3.22.

3.27. The degree of $G \circ F$ is the sum of the degrees of F and G. For the proof, use Problem 3.26.

3.29. Use the fact that the group $GL_2(\mathbb{R})$ of two by two invertible matrices has two connected components, cf. Appendix C.

3.30. Deform from the map to its linear approximation. See §D1.

3.31. See §19a for the local structure of general analytic mappings.

3.32. Compute the change in angle of F along the arcs between points in $F^{-1}(P')$.

4.6. If r were a retraction, $F(P) = -r(P)$ would have no fixed point.

4.7. Given a mapping f of Y to itself, consider the composite $i \circ f \circ r$, where i is the inclusion of Y in X and r is the retract.

4.8. (i). Note that (ii) and (iv) are homeomorphic.

4.11. Compare the restriction of f to S^1 with the identity mapping and the antipodal mapping, using the dog-on-a-leash theorem. Or look at fixed points of $x \mapsto \pm f(x)/|f(x)|$.

4.12. The unit vectors in the octant form a space homeomorphic to a disk, and, if no such vector is mapped to zero, then $F(P)/\|F(P)\|$ must have a fixed point.

4.13. Use the preceding exercise.

4.14. Look at the mapping $P \mapsto f(P) - P$, and use Exercise 4.11.

4.15. Show that f is homotopic to the antipodal map.

4.17. See Problem 3.23.

4.18. Map D^∞ to S^∞ by a formula $(a_0, a_1, \ldots) \mapsto (t, a_0, a_1, \ldots)$.

4.24. See the proof of Lemma 4.20.

4.27. See Lemma 4.21.

4.30. If not, do a spherical projection from a point not in the image.

4.31. If $f(P) \neq P$ for all P, f is homotopic to the antipodal map, while if $f(P) \neq P^*$ for all P, f is homotopic to the identity map.

4.38. Choose three arcs covering the circle without antipodal points, and look at their inverse images in the sphere.

4.39. Look at $A \cup B^*$, $B \cup C^*$, and $C \cup A^*$.

4.40. Tennis anyone?

5.4. If A is unbounded, the same proof shows that ω_P is exact. If A is bounded, the integral of ω_P around a large circle is nonzero.

5.13. Take $U = \mathbb{R}^2 \setminus A$, $V = \mathbb{R}^2 \setminus B$, and show that the image of δ and the kernel of δ each have dimension at least 1.

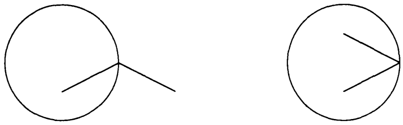
5.14. Divide the rectangle in half, with the intersection an interval. Argue as in Theorem 5.1, and use the fact $(*)$ to know about the first cohomology groups of the complements.

5.16. Write $X = A \cup B$, with A and B homeomorphic to circles, and $A \cap B$ a point or homeomorphic to an interval.

5.21. Induct on e. Let A be the union of the vertices and $e - 1$ of the edges, and let B be the other (closed) edge, and set $U = \mathbb{R}^2 \setminus A$ and $V = \mathbb{R}^2 \setminus B$, arguing separately the cases when B has one endpoint or two, and, when two, whether they are in the same component of A or not.

5.22. Analyze the connected components of the complement as the edges are added.

5.23. Take $V = \mathbb{R}^2 \setminus X$, so $U \cup V = \mathbb{R}^2$ and $U \cap V = U \setminus X$.

5.25. 

5.26. Look at the image of a circle around the band, and the image of its complement.

5.27. For example, the situation should look locally—via a homeomorphism—like the two axes crossing at the origin.

5.28. Both follow from Corollary 5.18.

6.5. If $\gamma = \Sigma n_i \gamma_i$, then $\gamma = \Sigma n_i(\partial \Gamma_i)$, where $\Gamma_i(t, s) = (1 - s) \cdot \gamma_i(t) + s \cdot P_0$, where P_0 is the point with respect to which U is starshaped.

6.12. Use Theorem 6.11. See §9a for generalizations.

6.14. If Γ: $[0, 1] \times [0, 1] \to U$, the boundary of $F \circ \Gamma$ is $F_*(\partial \Gamma)$.

6.17. See Proposition 7.5 and Problem 7.8.

6.18. See Problem 3.23.

6.24. (b) A point on a circle is a retract but not a deformation retract.

6.25. Use Exercise 6.20 and Proposition 6.23 to show that r_* is the inverse isomorphism to i_*.

6.27. If P' is a point not in X', use Tietze to extend F to a continuous mapping from an open neighborhood U of X to $\mathbb{R}^2 \setminus \{P'\}$. Apply Theorem 6.11.

6.28. The converse is false! For a counterexample, see Problem 13.28.

7.2. Consider $(\Re(x+iy)^n, \Im(x+iy)^n)$ and $(\Re(x+iy)^n, -\Im(x+iy)^n)$, where $\Re$ and $\Im$ denote the real and imaginary parts.

7.3. If γ is a path around P as usual, $H \circ \gamma$ gives a homotopy in $\mathbb{R}^2 \setminus \{0\}$ from $V_0 \circ \gamma$ to $V_1 \circ \gamma$.

7.4. In the first case, consider the homotopy

$$H(t,s) = s \cdot V(\gamma_r(t)) + (1-s) \cdot \rho(\gamma_r(t)) \cdot V(\gamma_r(t)),$$

$0 \le t \le 1$, $0 \le s \le 1$, where γ_r is a path around a small circle around P. For the second, compare V and $-V$.

7.8. If an infinite number of points P_i in Z have $W(\gamma,P_i) \neq 0$, such points lie in a bounded set, so they must have a limit point P. Since P cannot be in U, $W(\gamma,P) = 0$, and this contradicts Proposition 6.8.

7.9. Recall that f has a local maximum (resp. minimum) at P if the Hessian is positive and $(\partial^2 f/\partial x^2)(P) < 0$ (resp. $(\partial^2 f/\partial x^2)(P) > 0$); and f has a saddle point at P if the Hessian is negative. Use Problem 3.30.

7.13. Show that

$$(\varphi^* V)(x,y) = \begin{bmatrix} y^2 - x^2 & 2xy \\ -2xy & y^2 - x^2 \end{bmatrix} \cdot \begin{bmatrix} a + p(x,y) \\ b + q(x,y) \end{bmatrix},$$

with (a, b) a nonzero vector, and where $p(x,y)$ and $q(x,y)$ approach zero as $|x|$ and $|y|$ approach infinity. Restrict to a large circle, and do a homotopy (see Appendix D), to deform $\begin{bmatrix} a + p(x,y) \\ b + q(x,y) \end{bmatrix}$ first to $\begin{bmatrix} a \\ b \end{bmatrix}$, and then to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

7.16. See the first picture in this section.

7.17. Look at $V(P) = f(P) - (f(P) \cdot P)P$.

8.4. Thinking of a horizontal doughnut with g holes, put a source on top between each of the holes, and a sink directly under each source.

8.5. #peaks + #valleys − #passes = $2 - 2g$.

8.6. No.

8.10. Triangulate each of the polygons by putting a new vertex in its center.

8.12. Lift the vector field or triangulation to S^2.

8.13. 0.

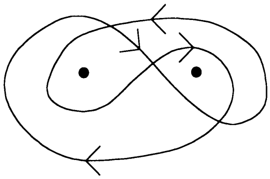
8.14. (b) A Klein bottle.

8.15. -1.

9.6. There is no homomorphism from $\mathbb{Z}^{n'}$ onto $\mathbb{Z}^n$ if $n > n'$, see Exercise C.14.

9.7. For (a), take a small circle γ_i around i for each $i \in \mathbb{N}$, and show that any closed 1-chain is homologous to a finite sum $\Sigma n_i \gamma_i$. The answer is the same for (b), since the spaces are homeomorphic, for example by the map $z \mapsto 1/z$ from $\mathbb{C} \setminus \{0\}$ to $\mathbb{C} \setminus \{0\}$.

9.9. The fact that the map is one-to-one follows from Theorem 6.11. For surjectivity, one can produce 1-cycles with arbitrary winding numbers around each K_i by Lemma 9.1, so the essential point is to show that the map from $H_1(U \setminus K)$ to $H_1 U$ is surjective. Take a grid so that no rectangle meets a point of K and a point not in U. If γ is a 1-cycle on U, we know γ is homologous to a sum of the form $\Sigma n_i \partial R_i$. Let $\gamma' = \Sigma_{R_i \cap K = \varnothing} n_i \partial R_i$. Then γ' is a 1-cycle on $U \setminus K$ that is homologous to γ on U. (See also Exercise 10.14.)

9.14. Consider the path  . See Exercise 11.14.

9.15. Use Corollary 9.12. If each γ_i is $\mathscr{C}^\infty$, so are all the constructions made in the proof that γ is a boundary.

9.16. Show that $\mathbb{R}^2 \setminus U$ is connected. Note that if V is any connected component of $\mathbb{R}^2 \setminus X$, then $\overline{V} \subset V \cup X$ and $\overline{V}$ meets X, so the union of any such V with X is connected.

9.17. For (a), take a subdivision and rectangles U_i as in the definition, but with the additional properties that each side of each U_i is of length at most 1, and the closure of U_i is contained in U. Let O_i be the point in the center of U_i. Given a point t_0, and an $\varepsilon > 0$, show that there is a $\delta > 0$ so that

$$|p(x,y,t) - p(x,y,t_0)| < \varepsilon/2n \quad \text{and} \quad |q(x,y,t) - q(x,y,t_0)| < \varepsilon/2n$$

for $|t - t_0| < \delta$. For each such t, let $f_{i,t}$ be the function on U_i so that $d(f_{i,t}) = \omega_t$ on U_i and such that $f_{i,t}(O_i) = 0$. Use the construction of Proposition 1.12, with the fact that the integrals are taken over segments of length at most $1/2$ (see Exercise B.8) to show that $|f_{i,t}(P) - f_{i,t_0}(P)| < \varepsilon/2n$ for all P in U_i, and for all $|t - t_0| < \delta$. Deduce that for $|t - t_0| < \delta$,

$$\begin{aligned} \left| \int_\gamma \omega_t - \int_\gamma \omega_{t_0} \right| &= \sum_{i=1}^{n} ((f_{i,t}(P_i) - f_{i,t_0}(P_i)) - (f_{i,t}(P_{i-1}) - f_{i,t_0}(P_{i-1})) \\ &\leq 2n \cdot (\varepsilon/2n) = \varepsilon . \end{aligned}$$

For (b), note that integral-valued locally constant functions are constant, and they vanish if they are small.

9.20. 53π.

9.21. Let P_i be any point in A_i, and set $\omega = \Sigma \mathfrak{p}_i \omega_{P_i}$, where ω_{P_i} is the 1-form $(1/2\pi)\omega_{P_i,\vartheta}$ that measures change in angle around P_i.

9.22. For (b), use the equation displayed after Corollary 9.19, with $(m_1, m_2) = (1, 0)$ and $(0, 1)$.

9.23. Approximate $u(a + \Delta x + i\Delta y) - u(a)$ by $(\partial u/\partial x)(a)\Delta x + (\partial u/\partial y)(a)\,\Delta y$, and similarly for v.

9.25. For (b), take γ_ε a circle of radius ε around the singularity, and let ε approach 0.

9.28. See Problem 7.8.

9.31. See Exercise 7.4.

9.35. Use the dog-on-a-leash theorem (Theorem 3.11).

10.5. On a path-connected space, 0-cycles of degree zero are boundaries.

10.14. Apply Mayer–Vietoris, with $V = \mathbb{R}^2 \setminus K$. Use Corollary 9.4 to calculate H_1V.

10.18. Identify S^2 with $\mathbb{R}^2 \cup \{\infty\}$. Suppose $X \setminus X \cap U = \{P, Q\}$, with P and Q in $\mathbb{R}^2$. Let $V = S^2 \setminus X$. Use Mayer–Vietoris to see that $U \cap V$ is disconnected exactly when $H_1U \to H_1(S^2 \setminus \{P, Q\})$ is zero, or equivalently, when $W(\gamma, P) = W(\gamma, Q)$ for all 1-cycles γ on U.

10.19. See Problem C.16.

10.20. Use Corollary 9.4 to compute H_1U, H_1V, and $H_1(U \cup V)$; and compute the kernel of ∂. For the last part, argue as in the proof of Theorem 5.11, using the inclusion of $H_0(U \cap V)$ in $H_0(U) \oplus H_0(V)$ to show that if points P_0 and P_1 are in different connected components of U and V, then they are in different connected components of $U \cap V$.

10.21. Use Exercise 10.14 with MV(iii) and MV(iv).

10.22. Apply Alexander's lemma, with the compact sets X and $\partial D \cup B$.

10.23. If not, find points P_n and Q_n within distance $1/n$ that cannot be so joined, and apply the preceding problem to a limiting point P.

10.24. Let $\varepsilon_n = 1/n$, and take corresponding $\delta_n > 0$ from the preceding problem. Take a sequence of points Q_n in the same component as Q, with the distance from Q_n to P at most $\delta_n/2$. Connect Q_n to Q_{n+1} by a path in a disk of radius ε_n. Join these paths together, with the nth path defined on a subinterval of length $1/2^n$.

10.25. Use Problem 9.9 with MV(iii) and MV(iv).

10.28. Take $V = \mathbb{R}^2 \setminus K$, and use the isomorphism described after Exercise 9.21.

10.30. If not, use grids to find a closed path γ in $\mathbb{R}^2 \setminus \partial U$ that has different winding numbers about two points of ∂U. Note that $\partial U \subset K$. Since γ is connected and does not meet $\partial U = \overline{U} \setminus U$, γ must be contained in U or in $\mathbb{R}^2 \setminus \overline{U}$. If γ is contained in U, then γ has the same winding number about all points in K, since K is connected and contained in $\mathbb{R}^2 \setminus U$; this contradicts the fact that $\partial U \subset K$. If γ does not meet $\overline{U}$, since $\overline{U}$ is connected, the winding number of γ is constant around points in $\overline{U}$, contracting the fact that $\partial U \subset \overline{U}$.

If K is closed and connected, the same is true. To see it, let $\overline{K}$ be the closure of K in $S^2 = \mathbb{R}^2 \cup \{\infty\}$, let P be a point in U, and apply the preceding case to $\overline{K} \subset S^2 \setminus \{P\} \cong R^2$.

11.1. For (ii) you can use the identity $\exp(x + iy) = e^x(\cos(y) + i\ \sin(y))$.

11.4. Show that the set in X where the cardinality is n is open and closed.

11.10. Write an open rectangle as an increasing union of closed rectangles.

11.11. For (a), when X is locally connected, the evenly covered neighborhoods N can be taken to be connected, and if $p^{-1}(N)$ is a disjoint union of open sets N_α, each mapping homeomorphically to N, then these N_α are the connected components of $p^{-1}(N)$. (b) follows, since Y' will be a union of those N_α that it meets.

11.13. For the triangulation, use the lifting propositions to lift any triangulation of S to a triangulation of S'.

11.15. For (b), for $n \in \mathbb{Z}$ and (r, ϑ) in the right half plane, the action is given by $n \cdot (r, \vartheta) = (r, \vartheta + 2\pi n)$.

11.22. The map from $X \times G$ to Y by $x \times g \mapsto g \cdot s(x)$ is a G-isomorphism.

11.23. Use the preceding exercise.

11.24. It's enough to look where the covering is trivial, as in the proof of Lemma 11.5.

11.28. Given y in Y, take disjoint neighborhoods U_g of $g \cdot y$, one for each g in G, and let V be the intersection of the open sets $g^{-1} \cdot U_g$.

11.39. Given such an automorphism φ of S, define an automorphism of the covering by the formula $y * \gamma \mapsto \varphi(y) * \gamma$ for any $y \in S$ and any path γ starting at x. Show that this is independent of choices.

12.1. Use the same formulas as in the preceding displays, but for the second variable s.

12.3. For (a),

$$H(t,s) = \begin{cases} \sigma\left(\dfrac{4t}{1+s}\right), & 0 \le t \le \tfrac{1}{4}(1+s), \\ \tau(4t-s-1), & \tfrac{1}{4}(1+s) \le t \le \tfrac{1}{4}(2+s), \\ \mu\left(\dfrac{4t-s-2}{2-s}\right), & \tfrac{1}{4}(2+s) \le t \le 1. \end{cases}$$

12.5. To compute $\pi_1(S^1,(1,0))$, see Problem 3.14, which is an easy consequence of the propositions in Chapter 11.

12.6. Use the homotopy $H(v,s) = sx + (1-s)v$.

12.8. Cut the path into a finite number pieces that map into hemispheres, say, and replace each piece by a homotopic arc with the same endpoints, for example an arc along a great circle.

12.10. If σ and τ are loops at e, consider the homotopy $H(t,s) = \sigma(t)\cdot\tau(s)$.

12.12. Map (t,s) to

$$\begin{cases} (0, 1-2t), & 0 \le t \le \tfrac{1}{2}(1-s), \\ \left(\dfrac{4t+2s-2}{3s+1}, s\right), & \tfrac{1}{2}(1-s) \le t \le \tfrac{1}{4}(s+3), \\ (1, 4t-3), & \tfrac{1}{4}(s+3) \le t \le 1. \end{cases}$$

Follow this by h to achieve the homotopy.

12.15. (i), (ii), (iv), and (v) are equivalent; (iii), (vi), and (viii) are equivalent; (vii) and (ix) are equivalent.

12.18. For S^1, let $H((x_1,x_2)\times s) =$ rotation by $s\cdot\pi$ acting on (x_1,x_2). For larger n, use the same formulas for each successive pair of the $n+1$ coordinates on $S^n \subset \mathbb{R}^{n+1}$.

12.20. Show that it has a circle SO(2) as a deformation retract.

13.10. Take $X \subset \mathbb{R}^2$ to be the union of the lines $L(n) = \{1/n\}\times[0,1]$, for all positive integers n, and three lines $L = [-1,1]\times\{1\}$, $M = \{-1\}\times[0,1]$, and $N = [-1,0]\times\{0\}$. Let x be the point $(0,0)$. Take two copies X_1 and X_2 of X, and denote by subscripts the corresponding lines and points in X_1 and X_2. Take Y to be the disjoint union of X_1 and X_2, topologized as usual except near the points x_1 and x_2. For a disk U of radius $\varepsilon < 1$ about $(0,0)$ in the plane, define a neighborhood $U(x_i)$ in X_i by

$$U(x_1) = (N_1 \cap U) \cup \bigcup_{n\,\text{odd}} (L(n)_1 \cap U) \cup \bigcup_{n\,\text{even}} (L(n)_2 \cap U),$$

$$U(x_2) = (N_2 \cap U) \cup \bigcup_{n\,\text{odd}} (L(n)_2 \cap U) \cup \bigcup_{n\,\text{even}} (L(n)_1 \cap U).$$

This space Y is connected, and the natural map from Y to X is a covering map.

13.12. The proof is exactly the same as for the theorem.

13.17. By Corollary 13.16, the answer to (a) is $\mathbb{Z}/n\mathbb{Z}$. For (b), the fundamental group is the group of translations of the plane described in Exercise 11.25. The subgroup in (c) is generated by a and b^2.

13.21. For (1), note that $\gamma \cdot \alpha$ is homotopic to $(\gamma \cdot \beta) \cdot (\beta^{-1} \cdot \alpha)$. For (3), if $\gamma \cdot \alpha$ is homotopic to $\gamma' \cdot \beta$, with α and β paths in N from z to w, then $\alpha \cdot \beta^{-1}$ is homotopic to ε_z in X, so γ is homotopic to $\gamma \cdot (\alpha \cdot \beta^{-1})$, so to $(\gamma \cdot \alpha) \cdot \beta^{-1}$, so to $(\gamma' \cdot \beta) \cdot \beta^{-1}$, so to $\gamma' \cdot (\beta \cdot \beta^{-1})$, and so to γ'.

13.22. Take Y to be the union of $\mathbb{R}$ with a copy X_n of a clamshell—but without its outer circle—attached at each point n in $\mathbb{Z} \subset \mathbb{R}$, and map Y to X by wrapping $\mathbb{R}$ around the outer circle of X once between each integer. Take $Z \rightarrow Y$ to be a covering that is nontrivial over a different circle of each X_n.

13.27. For $\Leftarrow$, write γ as a boundary on U, and apply r to both sides.

13.28. Let γ be a path which, on $[0, 1/2]$ first goes around C_1 counterclockwise, then C_2 counterclockwise, then C_1 clockwise, then C_2 clockwise. On $[1/2, 3/4]$ it does the same but using C_3 and C_4, and so on, on intervals of length $1/2^n$ using the circles C_{2n-1} and C_{2n}. For all k there are homomorphisms from the fundamental group to the free group F_k with k generators, obtained by using the first k circles (see Problem 13.25 or §14d). The image of $[\gamma]$ in F_{2n} is a commutator of n elements, but not of fewer than n elements. So $[\gamma]$ cannot be the commutator of any finite number of elements.

13.29. $(r, \vartheta) \mapsto \log(r) + i\vartheta$.

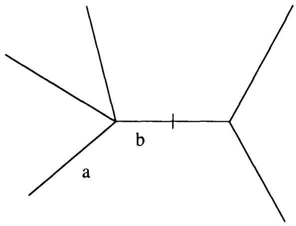
14.5. If Γ and Γ' were two such groups, the universal property for each would give maps from Γ to Γ' and from Γ' to Γ, and the two composites $\Gamma \rightarrow \Gamma' \rightarrow \Gamma$ and $\Gamma' \rightarrow \Gamma \rightarrow \Gamma'$ would be the identity maps by the uniqueness of such homomorphisms.

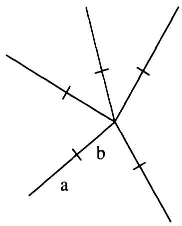
14.6. The subgroup of $\pi_1(X,x)$ that is generated by these images has the same universal property. It can be proved directly by subdividing the paths.

14.10. Take $G = \pi_1(U,x)$ and then $G = \pi_1(V,x)$.

14.13. See Problem 13.25 for the uniqueness.

14.15. If X' is X with an edge collapsed, map X' back to X as indicated:





The map is the identity except on edges adjacent to the vertex the edge is collapsed to. Send this vertex to the midpoint of the collapsing edge, and send half of each edge adjacent to this vertex to half of the edge that was collapsed, and spread the other half over the full edge. Check that the two composites are homotopic to the identity maps on X and X'.

14.17. For one point in a torus, the complement has a figure 8 as a deformation retract.

14.18. It is a free group on a countably infinite number of generators. Use Theorem 14.11.

14.19. The complement of an infinite discrete set in the plane is a covering space of the complement of a point in a torus.

14.20. The sphere with two handles can be obtained by joining the complements of disks in two tori. For another approach, see Chapter 17.

15.17. See Problem 9.7. The answer is the same for (b), and (c), see Problem 14.18.

15.19. See Chapter 24 for more general results.

16.5. Show that the set defined this way is open and closed.

16.6. With $\omega = df_\alpha$ on U_α, trivialize the covering over U_α, by mapping $p_\omega^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{R}$ by taking a germ f at P to $P \times (f(P) - f_\alpha(P))$. Check that $g_{\alpha\beta} = f_\alpha - f_\beta$ are transition functions for this covering. If $\omega' = \omega + dg$, let $f_\alpha' = f_\alpha + g$, and one obtains the same transition functions.

16.11. $H^0X \to H^0(X;\mathbb{R})$ comes from the fact that a locally constant function is a function on X with coboundary zero; $H^1X \to H^1(X;\mathbb{R})$ comes from the fact that a closed 1-form defines by integration a function on paths that is a 1-cocycle. It follows from Proposition 16.10 and Theorem 15.11 (and Exercise 15.18) that these maps are isomorphisms.

16.12. Use the ideas of Lemma 10.2. (Caution: Proposition 16.10 and Mayer–Vietoris for homology can be used directly for example when $G = \mathbb{R}$, but not in general.) See §24a for the general story.

16.13. Since the given covering is locally isomorphic to the trivial G-covering, it suffices to prove that p_T is a trivial covering when the given covering is the trivial G-covering. If the given covering is the projection from the product $X \times G \to X$, there is a canonical mapping

$$Y_T = ((X \times G) \times T)/G \to X \times T$$

determined by the map $\langle (x \times g) \times t \rangle \mapsto x \times g^{-1} \cdot t$. To see that this is well defined on orbits, note that, for h in G,

$$\begin{aligned} h \cdot \langle (x \times g) \times t \rangle &= \langle (x \times h \cdot g) \times h \cdot t \rangle \\ &\mapsto x \times (hg)^{-1} h \cdot t = x \times g^{-1} \cdot t. \end{aligned}$$

Check that this is a homeomorphism, with its inverse determined by sending $x \times t$ to $\langle (x \times e) \times t \rangle$.

16.14. For any y in Y and t in T, there is a unique element $\Phi(y, t)$ in T' so that

$$f(\langle y \times t \rangle) = \langle y \times \Phi(y, t) \rangle \qquad \text{for all } y \text{ in } Y \text{ and } t \text{ in } T.$$

For fixed t, the mapping $y \mapsto \Phi(y, t)$ from Y to T' is locally constant, since it is constant on each piece of $p^{-1}(N)$, for N an evenly covered set in X. But since Y is connected, a locally constant function is constant; therefore $\Phi(y, t) = \varphi(t)$ for some function $\varphi\colon T \to T'$. To see that φ is a map of G-sets, calculate:

$$\begin{aligned} \langle y \times \varphi(g \cdot t) \rangle &= f(\langle y \times g \cdot t \rangle) = f(\langle g^{-1} \cdot y \times t \rangle) \\ &= \langle g^{-1} \cdot y \times \varphi(t) \rangle = \langle y \times g \cdot \varphi(t) \rangle, \end{aligned}$$

from which the equation $\varphi(g \cdot t) = g \cdot \varphi(t)$ follows. By the definition, the mapping from Y_T to $Y_{T'}$ determined by φ is f.

16.17. There is a mapping from Y/H to $(Y \times G/H)/G$ that takes the H-orbit of a point y to the G-orbit of the point $y \times H$. The inverse mapping from $(Y \times G/H)/G \to Y/H$ is given by sending the G-orbit of $y \times gH$ to the H-orbit of $g^{-1} \cdot y$.

16.22. By Exercise 16.14, an isomorphism $f\colon Y(\psi_1) \to Y(\psi_2)$ is given by a map $\varphi\colon G' \to G'$ of G-sets. So

(i) $f(\langle z \times g' \rangle) = \langle z \times \varphi(g') \rangle$ for all $z \in Y$ and $g' \in G'$; and
(ii) $\varphi(g' \cdot \psi_1(g^{-1})) = \varphi(g') \cdot \psi_2(g^{-1})$ for all $g' \in G'$ and $g \in G$.

Since f preserves base points,

$$f(\langle y \times e \rangle) = \langle y \times \varphi(e) \rangle = \langle y \times e \rangle,$$

so we must have $\varphi(e) = e$. Since f is a mapping of G'-coverings, we must have

$$f(\langle z \times g' \rangle) = f(g' \cdot \langle z \times e \rangle) = g' \cdot f(\langle z \times e \rangle) = g' \cdot \langle z \times e \rangle = \langle z \times g' \rangle.$$

Therefore $\varphi(g') = g'$ for all g', and applying (ii) with $g' = e$ we see that $\psi_1(g^{-1}) = \psi_2(g^{-1})$ for all g in G, so $\psi_1 = \psi_2$.

16.24. Over U_α, where $p^{-1}(U_\alpha) \cong U_\alpha \times G$, identify $(p^{-1}(U_\alpha) \times T)/G$ with the product $U_\alpha \times T$ by $\langle x \times g \times t \rangle \mapsto x \times g^{-1} \cdot t$. The transition from $U_\alpha \times T$ to $U_\beta \times T$ is

$$x \times t \mapsto \langle x \times e \times t \rangle \mapsto \langle x \times g_{\alpha\beta}(x) \cdot e \times t \rangle \mapsto x \times g_{\alpha\beta}(x)^{-1} \cdot t = x \times g_{\beta\alpha}(x) \cdot t.$$

16.25. Consider first the case of a trivial covering $Y = X \times G$. Then

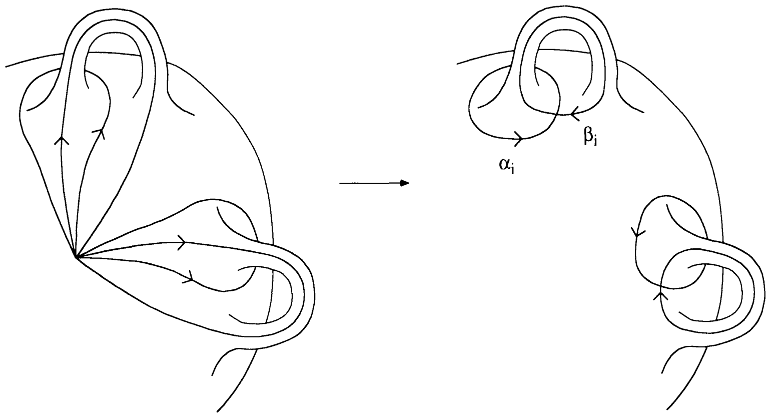
$$(Y \times G')/G = ((X \times G) \times G')/G \xrightarrow{\cong} X \times G', \quad x \times g \times g' \mapsto x \times g' \cdot \psi(g).$$

Choose trivializations $p^{-1}(U_\alpha) \cong U_\alpha \times G$ of the covering p, so that the resulting transitions are given by the cocycle $\{g_{\alpha\beta}\}$. Identify $p'^{-1}(U_\alpha)$ with $U_\alpha \times G'$ by the displayed isomorphism. The transition from $U_\alpha \times G'$ to $U_\beta \times G'$ (over $U_\alpha \cap U_\beta$) is

$$x \times g' \mapsto \langle x \times e \times g' \rangle \mapsto \langle x \times g_{\alpha\beta}(x) \cdot e \times g' \rangle \mapsto x \times g' \cdot \psi(g_{\alpha\beta}(x)).$$

16.28. The transition functions are given by the Jacobian determinants of the change of coordinate mappings.

17.10.



17.12. The fundamental group is the free group with $2g + n$ generators $a_1, b_1, \ldots, a_g, b_g, d_1, \ldots, d_n$, divided by the least normal subgroup containing

$$c_g = a_1 \cdot b_1 \cdot a_1^{-1} \cdot b_1^{-1} \cdot \ldots \cdot a_g \cdot b_g \cdot a_g^{-1} \cdot b_g^{-1} \cdot d_1 \cdot \ldots \cdot d_n.$$

The result is a free group on $2g + n - 1$ generators, if $n \geq 1$ (since one can write d_n in terms of the other generators).

18.3. These integers are determined by writing the class of γ in terms of the basis: $[\gamma] = \Sigma_{i=1}^{g}(m_i[a_i] + n_i[b_i])$.

18.4. If $\{\psi_\beta'\}$ is another partition of unity subordinate to another atlas of charts,

$$\sum_\beta\left(\iint_X \psi_\beta'\cdot\nu\right) = \sum_\beta\left(\sum_\alpha\left(\iint_X \psi_\alpha\cdot\psi_\beta'\cdot\nu\right)\right) = \sum_{\alpha,\beta}\left(\iint_X \psi_\alpha\cdot\psi_\beta'\cdot\nu\right)$$
$$= \sum_\alpha\left(\sum_\beta\left(\iint_X \psi_\beta'\cdot\psi_\alpha\cdot\nu\right)\right) = \sum_\alpha\left(\iint_X \psi_\alpha\cdot\nu\right).$$

The linearity is immediate from the definitions. For the opposite orientation, one can use the same charts but with the x and y axes interchanged, so in local coordinates the form ν is expressed as $-\nu_\alpha\, dy\, dx$, and all the integrals get replaced by their negatives.

18.7. Since the 1-forms α_i and β_i form a basis for H^1X, it suffices to prove these formulas when ω is one of the 1-forms α_i or β_i. Lemmas 18.1 and 18.6 imply these formulas. For example, (α_j, α_i) and $\int_{a_j}\alpha_i$ are both 0, as are (β_j, β_i) and $\int_{b_j}\beta_i$, and (α_j, β_i) and $\int_{a_j}\beta_i$ are both 1 if $i=j$, and 0 otherwise. Finally, $(\beta_j, \alpha_i) = -(\alpha_i, \beta_j)$, and $-(\alpha_i, \beta_j)$ and $\int_{b_j}\alpha_i$ are both -1 if $i=j$ and 0 otherwise.

18.8. By linearity, it is enough to do it for $\mu = \alpha_i$ and $\mu = \beta_i$, and it then follows from Exercise 18.7 and Lemma 18.1.

18.11. See Exercise 18.7 for (a).

18.12. Changing the a_i and b_i if necessary, one can assume they cross the annulus transversally. Calculate for $\mu = \alpha_i$ and β_i as above.

18.14. Use Problem 18.12.

18.15. It suffices to show that another choice of differentiable structure gives the same intersection numbers for pairs taken from basis elements a_i and b_j.

18.20. Use a partition of unity, as in Lemma 5.5.

18.22. If $H^2U = 0$ and $H^2V = 0$, it follows from this and the Mayer–Vietoris sequence in §16 that $H^2(U \cup V) = 0$. Use Exercise 18.18 and the fact that X can be built from rectangles, see Lemma 24.10.

18.25. One can realize the surface by removing h disks from a sphere, and gluing in h Moebius bands. Apply Mayer–Vietoris.

18.26. If f is a finite covering, and g is a function on Y, define $f_*(g)$ to be the function whose value at x is the sum of the values of g at the points of $f^{-1}(x)$. A similar definition works for forms.

19.1. $f'(z) = u_x + iv_x = v_y - iu_y$, and the Jacobian determinant is $u_xv_y - v_xu_y$.

19.4. Apply Riemann's theorem on removable singularities (Exercise 9.25).

19.6. For (d), multiply the function by a suitable $p_2(z)/p_1(z)$ so that the result has no poles, so is constant.

19.8. This follows from the fact that the map $z \mapsto z^e$ from D to D is proper, and the fact that only finitely many points are added.

19.17. (b) If $\lambda \cdot (\mathbb{Z} + \mathbb{Z}\tau') \subset \mathbb{Z} + \mathbb{Z}\tau$, $\lambda \cdot \tau' = a\tau + b$, $\lambda \cdot 1 = c\tau + d$; the determinant is ± 1 exactly when $\lambda \cdot (\mathbb{Z} + \mathbb{Z}\tau') = \mathbb{Z} + \mathbb{Z}\tau$, and it is positive if τ and τ' are both in the upper half plane.

20.5. To prove that C is a Riemann surface near such a point, show that one or the other projection to an axis is a local isomorphism.

20.6. The genus in each case is: (i) $(m-2)/2$ if m is even, $(m-1)/2$ if m is odd; (ii) 0; (iii) 4; (iv) 1; and (v) $(m-1)(m-2)/2$.

20.7. (c) With the notation of Exercise 19.12, the covering is given by assigning each σ_i to the unique transposition $(1\ \ 2)$.

20.12. The assumptions in (ii) guarantee that X has n distinct points over ∞, all therefore with ramification index 1. To see this, set $z' = 1/z$ and $w' = w/z$; they satisfy an equation $G(z', w') = \Sigma_{i=0}^{n} b_i(z')(w')^{n-1} = 0$, where $b_i(z') = (z')^i a_i(1/z')$ is a polynomial in z' with constant term λ_i; the n roots to the equation $\Sigma_{i=0}^{n} \lambda_i t^{n-i} = 0$ give n points on X over ∞.

Consider the zeros and poles of the meromorphic function h. To see what happens at the points over ∞, make the change of coordinates as above. A calculation shows that $h = (z')^{1-n} G_{w'}(z', w')$, so h must have pole of order $n-1$ at each of these n points. At the other points of X, i.e., the points of C, h has no poles, so by Corollary 19.5 the sum of the orders of zeros of h at the points of C must be $n(n-1)$. By Exercise 20.9, this sum is the sum of $e(P) - 1$ over all ramification points of the mapping $z: X \to S^2$. Apply Riemann–Hurwitz to prove (c).

The space of polynomials of degree at most $n-3$ in Z and W has a basis the monomials $Z^i W^j$ with $i + j \leq n-3$, and the number of these is

$$(n-2) + (n-3) + \ldots + 2 + 1 = (n-1)(n-2)/2.$$

If we verify that $g \cdot dz/h$ is a holomorphic 1-form for each $g = z^i w^j$, it follows that we have produced g_X holomorphic 1-forms, showing at once that these are all of the holomorphic 1-forms, and that their dimension is g_X. At the points in C, the form dz/h is holomorphic, as follows from the equation

$$0 = d(F(z,w)) = F_z(z,w)\,dz + F_w(z,w)\,dw,$$

so $dz/h = -dw/F_z(z,w)$, and one of the denominators is nonzero at each point of C. At infinity, since $dz = -(z')^{-2} dz'$,

$$\begin{aligned} z^i w^j \frac{dz}{F_w(z,w)} &= (z')^{-i} \left(\frac{w'}{z'}\right)^j \frac{-(z')^{-2}\,dz'}{(z')^{1-n} G_{w'}(z', w')} \\ &= -(z')^{n-3-i-j} (w')^j \frac{dz'}{G_{w'}(z', w')}, \end{aligned}$$

from which we see that these 1-forms are also holomorphic at points over ∞.

20.13. The proof is essentially the same, except that $F_w(z, w)$ vanishes at the 2δ points of X lying over the nodes, as well as at the branch points. Note that the space of polynomials of degree at most $n-3$ vanishing at δ points always has dimension at least $(n-1)(n-2)/2-\delta$, so the construction produces at least g_X independent holomorphic 1-forms. But since $\dim(\Omega^{1,0}) \leq g$, the inequalities are equalities—which shows, in fact, that conditions to vanish at the nodes are all independent.

20.16. Compare the sum of the orders of $f^*\omega$ with those of ω, for ω a meromorphic 1-form on Y.

20.17. See Exercise 9.26.

20.19. For (a), since the residue is linear, it is enough to prove it when $\omega = (z-a)^m dz$, $a \in \mathbb{C}$, $m \in \mathbb{Z}$. (b) can be reduced to a local calculation, over a disk in S^2, where one has explicit formulas for the map z.

20.24. The point is that, up to periods, integrating from P to Q and then from Q to R is the same as integrating from P to R.

21.2. Write $E = D + Q_1 + \ldots + Q_r$ and apply Lemma 21.1 r times.

21.13. Define the adele $\mathbf{f}$ to be f_i at P_i and 0 elsewhere. Take D of large degree so that $M + R(D) = R$, and with $\mathrm{ord}_{P_i}(D) \geq m_i$ for all i. There is therefore an f in M so that $f - \mathbf{f}$ is in $R(D)$.

21.20. For (a), if points $P_1, \ldots, P_k$ have been found so that $\dim(\Omega(P_1 + \ldots + P_k)) = g - k$, and $k < g$, take any nonzero ω in $\Omega(P_1 + \ldots + P_k)$, and let P_{k+1} be any point which is not a zero of ω. For (b), change the last P_g to be a zero of $\omega \neq 0$ in $\Omega(P_1 + \ldots + P_{g-1})$. For points as in (b), Riemann–Roch implies that $\dim(L(P_1 + \ldots + P_g)) \geq 2$, so there is a nonconstant function with at most g poles.

21.21. Multiplying φ by a scalar, we may arrange so the residues of φ at P and Q are as stated. This φ is unique up to adding a holomorphic ω. Take a_j, b_j, ω_j as in §20d, with the a_j and b_j not passing through P or Q. Use Corollary 20.22 to show that there are unique complex numbers λ_j so that the integral of $\varphi - \Sigma\lambda_j\omega_j$ over the cycles a_k and b_k are all purely imaginary.

21.22. Take φ as in the preceding exercise, and take u to be an integral of the real part of φ.

21.23. An element in $\Omega(-2P)$ that is not in $\Omega(0)$ must have a double pole, since it cannot have a residue at P. Multiply by a scalar to get $\varphi + dz/z^2$ holomorphic near P, change φ by a holomorphic 1-form to get all its periods purely imaginary, and integrate the real part of φ to get u.

21.24. Consider the sequence $\mathbb{C} = L(0) \subset L(P_1) \subset L(P_1 + P_2) \subset \ldots$, with the dimensions going up by 0 or 1 at each step, reaching dimension g when $k = 2g - 1$.

21.25. Apply Riemann–Roch, with $E = K - D$.

21.27. The holomorphic differentials have the form $[(a + bz + cw)/w^3]\,dz$, with a, b, $c \in \mathbb{C}$. Check that no such form vanishes to order 2 at any point of X.

22.1. See the proof of Green's theorem in Appendix B. For the correct signs, see §23a.

22.2. Extend the definition of integral over continuous paths as in the plane, and use the same arguments as in the planar case.

22.3. 4π.

22.4. If $F\colon S^2 \to \mathbb{R}^3 \setminus \{0\}$, define $W(F, 0)$ to be $(1/4\pi) \iint_{F \circ \Gamma} \omega$.

22.6. Use Van Kampen, with one open set a ball around the missing point.

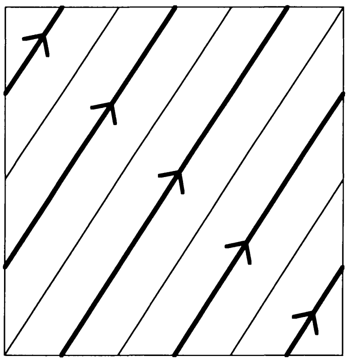
22.7. A homotopy is $H((z, w) \times s) = \left(sz, \sqrt{1 - s^2|z|^2}\,\dfrac{w}{|w|}\right)$.

22.9. To show the map is surjective, if (z, w) is in K, let $z' = -z/a$ and $w' = w/b$, and verify that $(z')^3 = (w')^2$ and $|z'| = |w'| = 1$.

22.11. For example, with $\rho(t) = 1 - t$, one may take

$$\lambda = \tfrac{1}{2}\left(-\rho^2 + \sqrt{4 + 4\rho^3 + \rho^4}\right) \quad \text{and} \quad \mu = \sqrt{\lambda - \rho}\,.$$

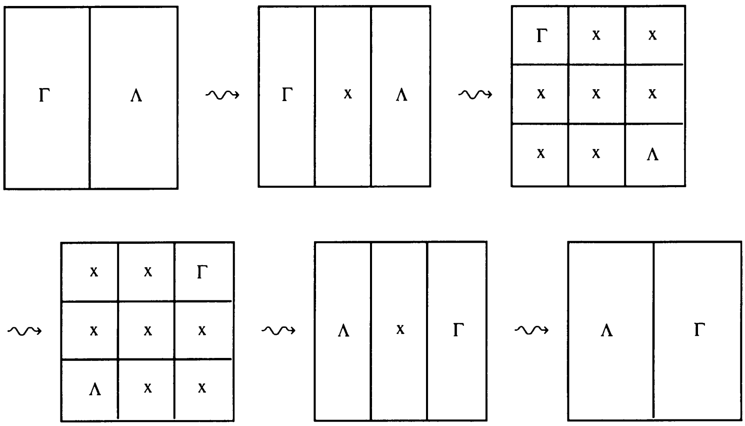
22.13. The generators for the fundamental groups of $A \setminus K$ and $B \setminus K$ are the circles around the middles of the solid tori. The generator for the fundamental group of $T \setminus K$ is a path as indicated:



To appeal to Van Kampen, A and B must be replaced by open neighborhoods of which they are retracts.

22.14. For (a), construct a mapping from $[0, 1]^k$ to S^k that maps the boundary to s_0, and is a homeomorphism from the interior of the cube to the complement of s_0. For (d), show that a map from S^k to S^n, with $k<n$, is homotopic to one that misses the south pole, say by triangulating S^k into small simplices, and approximating the map by a "simplicial" map. See, e.g., Hilton (1961).

22.16. Find homotopies that stretch and slide between the maps indicated in the diagram:



For example, the second homotopy maps $(t_1, \ldots, t_k, s)$ to

$$\begin{cases} \Gamma(3t_1, (3t_2 - 2s)/(3-2s), t_3, \ldots, t_k), & 0 \le t_1 \le 1/3, 2s/3 \le t_2 \le 1, \\ \Lambda(3t_1 - 1, 3t_2/(3-2s), t_3, \ldots, t_k), & 2/3 \le t_1 \le 1, 0 \le t_1 \le 1 - 2s/3, \\ x, & \text{otherwise}. \end{cases}$$

The third slides the squares around clockwise.

22.22. Details can be found in many texts, e.g., Bott and Tu (1980).

22.23. The form is closed by calculation, and it is not exact by Stokes theorem, since its integral over S^{n-1} is not zero.

22.27. See Bott and Tu (1980).

23.1. This is formal, using the identities

$$\partial_i^s \circ \partial_j^{s'} = \partial_{j-1}^{s'} \circ \partial_i^s \quad \text{for} \quad i<j,$$

and s and s' each taking values 0 and 1. The signs cancel because of the shift in subscript from j to $j-1$.

23.5. Check that $(\partial_i^0 - \partial_i^1) \circ S = S \circ (\partial_i^0 - \partial_i^1)$.

23.15. Induct on p, and use Mayer–Vietoris for $U = U_1 \cup \ldots \cup U_{p-1}$ and

$V = U_p$. Note that $U \cap V = (U_1 \cap V) \cup \ldots \cup (U_{p-1} \cap V)$, so the inductive hypotheses imply to U, V, and $U \cap V$.

23.17. For (d), argue by induction, using Mayer–Vietoris. If $g: S^{n-1} \to S^{n-1}$ has degree d, show that $f: S^n \to S^n$ given by the formula

$$f(x_1, \ldots, x_{n+1}) = (g(x_1, \ldots, x_n), x_{n+1})$$

also has degree d.

23.19. For (c), use Problem C.3 and the n-dimensional version of Lemma B.9; see the proof of Claim D.3.

23.22. Induct on n, comparing the action of the antipodal map with the Mayer–Vietoris isomorphism.

23.23. For (a), such a vector field gives a map $f: S^n \to S^n$ that has no point P mapped to P or to $-P$. Such a map is homotopic to the identity and to the antipodal map. Use the preceding problem. For (b), consider the mapping $(x_1, x_2, \ldots, x_{n+1}) \mapsto (x_2, -x_1, \ldots, x_{n+1}, -x_n)$.

23.26. For (a), regard $S^m \subset S^n$, and note that a map $f: S^n \to S^n$ that is not surjective has degree zero. The other proofs are essentially the same as in Chapter 4.

23.27. In the situation of (a), there is a homotopy from f to a map g to which Theorem 23.24 applies, given by homotopic to the map g given by

$$H(P \times s) = \frac{f(P) - sf(P^*)}{\|f(P) - sf(P^*)\|}.$$

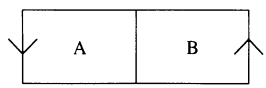
For (b) show similarly that f is homotopic to a map g with $g(P^*) = g(P)$ for all P. (c) Any automorphism g without fixed points has degree -1, so if g and h are nontrivial automorphisms, since $g \cdot h$ has degree 1, $g \cdot h$ must be the identity.

23.28. Lift to a map from S^n to S^n and apply Problem 23.27.

23.31. If $m = n$, $H_k(S^m \times S^n)$ is $\mathbb{Z}$ for $k = 0$ and $k = n + m$, and $\mathbb{Z} \oplus \mathbb{Z}$ if $k = n$, and 0 for other k. If $m \neq n$, the answer is $\mathbb{Z}$ for $k = 0$, m, n, and $m + n$, and 0 for other k. For the proof, induct on n, using $S^m \times U$ and $S^m \times V$, with U and V as before.

23.36. Identify $\mathbb{R}^n$ with the complement of a point P in S^n, and use Mayer–Vietoris for this open set and a small neighborhood of P.

23.37. See Proposition 5.17 and Corollary 5.18.

23.39. Cut the band in half: 

. Look at the exact sequence

$$0 \to H_2(\mathbb{R}^3 \setminus A) \oplus H_2(\mathbb{R}^3 \setminus B) \to H_2(\mathbb{R}^3 \setminus A \cap B) \to H_1(\mathbb{R}^3 \setminus X) \to 0,$$

and the maps to the terms in the corresponding sequence with A and B replaced by $A \cap Y$ and $B \cap Y$, and X replaced by Y.

23.41. Use n-dimensional grids, and generalize the arguments of Chapters 6 and 9.

24.5. The fact that C'' is free abelian means that one can find a subgroup $\widetilde{C}''$ of C that maps isomorphically onto C'', and then C is the direct sum of $\widetilde{C}''$ and the image of C'; and $\mathrm{Hom}(-, G)$ preserves direct sums.

24.7. This amounts to an identity on I^k, which is proved just as in the case of Green's theorem for a rectangle by a calculation using Fubini's theorem, as in Proposition B.6.

24.9. This is the higher-dimensional version of reparametrization for paths. Here, for any k-cube Γ, $S \circ A(\Gamma) - \Gamma = \partial\Lambda$, where

$$\Lambda(s, t_1, \ldots, t_k) = \Gamma(s \cdot \alpha(2t_1) + (1 - s) \cdot t_1, \ldots, s \cdot \alpha(2t_k) + (1 - s) \cdot t_k),$$

noting that the other terms in the boundary are degenerate. Defining S_p and R_p by the same formulas as in §23b, one calculates that

$$S^p - 1 = \partial \circ R_p + R_p \circ \partial + (S \circ A - I) \circ S_p,$$

and the rest of the proof is the same as before.

24.14. This is clear except where the coboundary and dual of the boundary are involved. Let $\omega = \omega_1 - \omega_2$ be a closed $(k-1)$-form representing a class $[\omega]$ in $H^{k-1}(U \cap V)$, with ω_1 and ω_2 $(k-1)$-forms on U and V, and let $z = c_1 + c_2$ represent a class $[z]$ in $H_k^\infty(U \cup V)$, where c_1 and c_2 are chains on U and V respectively. Then

$$\delta([\omega])([z]) = \int_{c_1} d\omega_1 + \int_{c_2} d\omega_2 = \int_{\partial c_1} \omega_1 + \int_{\partial c_2} \omega_2 = [\omega](\partial([z])).$$

24.15. See Problem 23.41.

24.16. Use Mayer–Vietoris and induct on the number of open sets in the cover.

24.21. Set $(p_*\omega)(x) = \omega(y_1) + \omega(y_2)$ where $p^{-1}(x) = \{y_1, y_2\}$. For (c), note that $\int_{\tilde{X}} p^*\omega = 0$ for any n-form ω with compact support on X.

24.23. For (a), use $H^kX \cong (H^{n-k}X)^* \cong ((H^kX)^*)^*$, and the general fact that if V is a vector space isomorphic to $(V^*)^*$, then V must be finite dimensional. For (b), use the linear algebra fact that a finite-dimensional vector space with a nondegenerate skew-symmetric form must be even dimensional.

24.24. Since the spaces are finite dimensional, $H_c^{n-p}X$ is isomorphic to $(H^pX)^*$.

24.25. This follows from the fact that all cubes Γ have compact image in X. (Compare the special case used in the argument in §5c.)

24.26. (b) follows from the definitions, just as for homology.

24.27. Use Exercise 24.5.

24.36. Consider for each n the exact sequences $0 \to Z_n \to C_n \to B_{n-1} \to 0$ and $0 \to B_n \to Z_n \to H_n \to 0$, and use Problem C.16.

24.37. For (c), fix an ordering on $N(\mathfrak{U})$. For each vertex v, choose a point c_v in U_v, regarded as a 0-chain on U_v. If (v_0, v_1) is a 1-simplex, then, since $U_{v_0} \cup U_{v_1}$ is connected, there is a path from v_0 to v_1 in $U_{v_0} \cup U_{v_1}$, which determines a 1-chain $c_{(v_0,v_1)}$. Construct, by induction on n, for each n-simplex $(v_0, \ldots, v_n)$, a chain $c_{(v_0,\ldots,v_n)}$ in $C_n(U_{v_0} \cup \ldots \cup U_{v_n})$, such that

$$\partial(c_{(v_0,\ldots,v_n)}) = \sum_{i=0}^{n} (-1)^i c_{(v_0,\ldots,\hat{v}_i,\ldots,v_n)} .$$

The existence of $c_{(v_0,\ldots,v_n)}$ follows from the fact that the right side is an $(n-1)$-cycle on $U_{v_0} \cup \ldots \cup U_{v_n}$, and $H_{n-1}(U_{v_0} \cup \ldots \cup U_{v_n}) = 0$ by Exercise 23.15(b). This gives a map $C_*(N(\mathfrak{U})) \to C_*X$. To see that the resulting map on homology is an isomorphism, induct on the number of open sets; take any v and construct subcomplexes L_1 and L_2 as in the proof of Proposition 24.33, and compare the corresponding Mayer–Vietoris sequence for L_1 and L_2 with that of the covering of X by U_v and $\cup_{v' \neq v} U_{v'}$.

A.6. If not, each point has a neighborhood meeting only finitely many, and K would be contained in a finite union of such neighborhoods.

A.8. Without loss of generality, one may assume K contains a neighborhood of the origin. Map ∂K to S^{n-1} by mapping P to $P/\|P\|$. This is continuous and bijective, so a homeomorphism. Let $f\colon S^{n-1} \to \partial K$ be the inverse map. Define $F\colon D^n \to K$ by $F(0) = 0$, and $F(P) = \|P\| \cdot f(P/\|P\|)$ for $P \neq 0$. Then F is continuous and bijective, so a homeomorphism.

A.16. A connected and locally path-connected space is path-connected, as seen by showing that the set of points that can be connected to a given point by a path is open and closed.

B.12. Find a countable covering of U and so that each open set in the covering is contained in some U_α, and so that any point has a neighborhood that only meets finitely many of the open sets.

B.14. With g as in Step (2), let $h(x) = \int_0^x g(t)\,dt / \int_0^1 g(t)\,dt$, and set $\psi(x,y) = h(r - r_1)/r_2 - r_1)$, where $r = \|(x,y)\|$.

C.7. If $\{e_\alpha\}$ is a basis for F, and $\varphi(\tilde{e}_\alpha) = e_\alpha$, these $\tilde{e}_\alpha$ generate a free abelian subgroup $\tilde{F}$ of A, and A is the direct sum of $\mathrm{Ker}(\varphi)$ and $\tilde{F}$ by Exercise C.5.

C.15. Let α be the map from A to B, β the map from B to C, and let C' be the image of β, $\beta'\colon B \to C'$ the induced surjection. If f in $\mathrm{Hom}(B, \mathbb{R})$ maps to 0 in $\mathrm{Hom}(A, \mathbb{R})$, f vanishes on the image of α. Since $B/\mathrm{Image}(\alpha) \cong C'$, there is a homomorphism g' from C' to $\mathbb{R}$ such that $g' \circ \beta' = f$. By the lemma, there is a homomorphism g from C to $\mathbb{R}$ that restricts to g' on C'. Then $g \circ \beta = f$, which means that g in $\mathrm{Hom}(C, \mathbb{R})$ maps to f.

C.16. Let $B = A_2/A_1$, and show that there are exact sequences

$$0 \to A_1 \to A_2 \to B \to 0 \quad \text{and} \quad 0 \to B \to A_3 \to \ . \ . \ . \to A_n \to 0 .$$

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191 LANG. Fundamentals of Differential Geometry.
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195 NATHANSON. Elementary Methods in Number Theory.
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198 ROBERT. A Course in p-adic Analysis.
199 HEDENMALM/KORENBLUM/ZHU. Theory of Bergman Spaces.
200 BAO/CHERN/SHEN. An Introduction to Riemann–Finsler Geometry.
201 HINDRY/SILVERMAN. Diophantine Geometry.